Problem 1: Let $\text{Mat}(n, \mathbb{R})$ denote the set of $n \times n$ real matrices, which we can identify with the Euclidean space $\mathbb{R}^{n^2}$. Explain why the determinant function
$$\det : \text{Mat}(n, \mathbb{R}) \to \mathbb{R}$$
is continuous.

Solution: Given an $n \times n$ matrix $x = (x_{i,j})_{1 \leq i,j \leq n}$, we have that
$$\det(x) = \sum_{\pi \in S_n} \epsilon_{\pi} x_{1,\pi_1} \cdots x_{n,\pi_n}.$$Here $S_n$ is the collection of all permutations $\pi = \pi_1 \pi_2 \cdots \pi_n$ of $1, 2, \ldots, n$ and the number $\epsilon_{\pi} \in \pm 1$ is the sign of the permutation $\pi$. This formula shows that $\det(x)$ is a polynomial function of the entries of the matrix $x$, and any polynomial function is continuous (with respect to the Euclidean topology).

Problem 2: Let $\text{GL}_n(\mathbb{R}) \subset \text{Mat}(n, \mathbb{R})$ be the subset of invertible $n \times n$ matrices. Prove that $\text{GL}_n(\mathbb{R})$ is open. (So, if a matrix $A$ is invertible, there exists $\epsilon > 0$ such that altering any entry of $A$ by at most $\epsilon$ results in an invertible matrix.)

Solution: Consider the determinant function $\det : \text{Mat}(n, \mathbb{R}) \to \mathbb{R}$ which sends any matrix to its determinant. It is a fact of linear algebra that, given $A \in \text{Mat}(n, \mathbb{R})$, we have that $A$ is invertible if and only if $\det(A) \neq 0$. Said differently, this means that $\text{GL}_n(\mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$. Since $\mathbb{R} - \{0\}$ is an open subset of $\mathbb{R}$ and $\det$ is continuous (by Problem 1), we conclude that $\text{GL}_n(\mathbb{R})$ is open in $\text{Mat}(n, \mathbb{R})$.

Problem 3: A subset $D$ of a space $X$ is called dense if $\overline{D} = X$. Let $X$ and $Y$ be spaces with $Y$ Hausdorff. Let $D \subset X$ be a dense subset. Let
$$f, g : X \to Y$$
be continuous functions such that $f(x) = g(x)$ for all $x \in D$. Prove that $f = g$.

Solution: Suppose $f \neq g$. Then there exists $x \in X$ such that $f(x) \neq g(x)$. Since $Y$ is Hausdorff, we can find neighborhoods $U$ of $f(x)$ and $V$ of $g(x)$ in $Y$ such that $U \cap V = \emptyset$. Since $f$ is continuous, we get that $f^{-1}(U)$ is a neighborhood of $x$ in $X$. Since $g$ is continuous, we get that $g^{-1}(V)$ is a neighborhood of $x$ in $X$. Therefore, we have that $W := f^{-1}(U) \cap g^{-1}(V)$ is a neighborhood of $x$ in $X$. Moreover, for any $x' \in W$, the condition $U \cap V = \emptyset$ forces $f(x') \neq g(x')$. We conclude that $W \cap D = \emptyset$, so that $x \notin \overline{D}$. This contradicts the density of $D$.

Problem 4: Prove that the conclusion of the last problem need not hold if $Y$ is not Hausdorff.
Solution: Let $X = \mathbb{R}$ with the standard topology and let $Y = \{a, b\}$ be a two-point space with the indiscrete topology. Define $f : X \to Y$ by $f(x) = a$ for all $x \in X$. Define $g : X \to Y$ by

$$g(x) = \begin{cases} a & x \in \mathbb{Q} \\ b & x \not\in \mathbb{Q}. \end{cases}$$

Both $f$ and $g$ are continuous because $Y$ has the indiscrete topology. We have that $f(x) = g(x)$ for $x \in \mathbb{Q}$, and we have $\overline{\mathbb{Q}} = X$. However, we have $f \neq g$.

Problem 5: Look up “stereographic projection”. Let $S^n$ be the $n$-sphere and let $N = (1, 0, \ldots, 0)$ be the ‘north pole’. Prove that $\mathbb{R}^n$ is homeomorphic to $S^n - \{N\}$.

Solution: As usual, we make the identification

$$S^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}.$$

Define a function $f : S^n - \{N\} \to \mathbb{R}^n$ by the formula

$$f(x_0, x_1, \ldots, x_n) := \left(\frac{x_1}{1-x_0}, \frac{x_2}{1-x_0}, \ldots, \frac{x_n}{1-x_0}\right).$$

Observe that $f$ is actually well defined because $-1 < x_0 < 1$ for any point on $S^n$ other than $N$. Clearly $f$ is continuous.

Define another function $g : \mathbb{R}^n \to S^n - \{N\}$ by the formula

$$g(y_1, \ldots, y_n) := \left(\frac{-1 + y_1^2 + \cdots + y_n^2}{1 + y_1^2 + \cdots + y_n^2}, \frac{2y_1}{1 + y_1^2 + \cdots + y_n^2}, \ldots, \frac{2y_n}{1 + y_1^2 + \cdots + y_n^2}\right).$$

To see that $g$ is well defined, notice that $(-1 + y_1^2 + \cdots + y_n^2)^2 + (2y_1)^2 + \cdots + (2y_n)^2 = (1 + y_1^2 + \cdots + y_n^2)^2$, so that $g$ actually maps into $S^n$. Moreover, since $-1 + y_1^2 + \cdots + y_n^2 \neq 1 + y_1^2 + \cdots + y_n^2$ for any $(y_1, \ldots, y_n) \in \mathbb{R}^n$, we have that $N$ is not in the image of $g$. Clearly $g$ is continuous.

Now we check that $f$ and $g$ are mutually inverse mappings. Let $(y_1, \ldots, y_n) \in \mathbb{R}^n$. We have that

$$f(g(y_1, \ldots, y_n)) = f\left(\frac{-1 + y_1^2 + \cdots + y_n^2}{1 + y_1^2 + \cdots + y_n^2}, \frac{2y_1}{1 + y_1^2 + \cdots + y_n^2}, \ldots, \frac{2y_n}{1 + y_1^2 + \cdots + y_n^2}\right) = (y_1, \ldots, y_n).$$
Let \((x_0, x_1, \ldots, x_n) \in S^n - \{N\}\). We have that
\[
g(f(x_0, x_1, \ldots, x_n)) = g\left(\frac{x_1}{1 - x_0}, \frac{x_2}{1 - x_0}, \ldots, \frac{x_n}{1 - x_0}\right)
\]
\[
= \left(\frac{1 - x_0}{2}, \frac{2x_0}{2}, \frac{1 - x_0}{2}, \ldots, \frac{1 - x_0}{2}, \frac{2x_n}{2}\right)
\]
\[
= (x_0, x_1, \ldots, x_n).
\]
In the second equality we used the fact that \(x_0^2 + x_1^2 + \cdots + x_n^2 = 1\), so that we have
\[
\frac{1}{(1-x_0)^2}(x_1^2 + \cdots + x_n^2) = \frac{1+x_0}{1-x_0}.
\]
We conclude that \(f\) and \(g\) are mutually inverse continuous mappings, and thus mutually inverse homeomorphisms. Therefore, we get that \(\mathbb{R}^n\) and \(S^n - \{N\}\) are homeomorphic, as desired.

**Problem 6:** Let \(A \subset \mathbb{R}^2\) be the annulus
\[
A = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}.
\]
Prove that \(A\) is homeomorphic to \(S^1 \times I\), where \(I = [0, 1]\) is the unit interval.

**Solution:** We identify \(S^1 \times I = \{(u, v, t) \in \mathbb{R}^3 : u^2 + v^2 = 1, 0 \leq t \leq 1\}\).

Define a function \(f : A \rightarrow S^1 \times I\) by the formula
\[
f(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, \sqrt{x^2 + y^2} - 1\right).
\]
Clearly \(f\) is well defined and continuous.

Define a function \(g : S^1 \times I \rightarrow A\) by the formula
\[
g(u, v, t) = ((1 + t)u, (1 + t)v).
\]
Clearly \(g\) is well defined and continuous.

We check that \(f\) and \(g\) are mutually inverse mappings. Let \((u, v, t) \in S^1 \times I\). Then
\[
f(g(u, v, t)) = f((1 + t)u, (1 + t)v)
\]
\[
= \left(\frac{(1+t)u}{1+t}, \frac{(1+t)v}{1+t}, (1 + t) - 1\right)
\]
\[
= (u, v, t),
\]
where the second equality used \(u^2 + v^2 = 1\).

Let \((x, y) \in A\). We have that
\[
g(f(x, y)) = g\left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, \sqrt{x^2 + y^2} - 1\right)
\]
\[
= (x, y).
\]
We conclude that $f$ and $g$ are mutually inverse continuous mappings, and hence mutually inverse homeomorphisms. Therefore, we get that $A$ is homeomorphic to $S^1 \times I$.

**Problem 7:** Let $f : A \to B$ and $g : C \to D$ be continuous functions. Define a new function

$$f \times g : A \times C \to B \times D$$

by

$$f \times g : a \times c \mapsto f(a) \times g(c).$$

Prove that $f \times g$ is a continuous function.

**Solution:** Let $U \times V \subset B \times D$ be a typical basic open set, where $U \subset B$ is open and $V \subset D$ is open. We compute

$$(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V).$$

Since $f$ is continuous, we have that $f^{-1}(U)$ is open in $A$. Since $g$ is continuous, we have that $g^{-1}(V)$ is open in $C$. We conclude that $f^{-1}(U) \times g^{-1}(V)$ is open in $A \times C$. Therefore, the map $f \times g$ is continuous.

**Problem 8:** Let $f : X \to Y$ be a function between topological spaces. If $X$ is discrete, prove that $f$ is continuous. If $Y$ is indiscrete, prove that $f$ is continuous.

**Solution:** Suppose $X$ is discrete. Let $U \subset Y$ be open. Then $f^{-1}(U) \subset X$ is open because every subset of $X$ is open in $X$. Therefore, $f$ is continuous.

Suppose $Y$ is indiscrete. The only open sets in $Y$ are $\emptyset$ and $Y$. We have $f^{-1}(Y) = X$ and $f^{-1}(\emptyset) = \emptyset$, which are both open in $X$. Therefore, $f$ is continuous.

**Problem 9:** Let $f : X \to Y$ be a continuous function. The **graph** of $f$ is the subspace $\Gamma_f \subset X \times Y$ given by

$$\Gamma_f := \{(x, f(x)) : x \in X\}.$$ 

Prove that $X$ is homeomorphic to $\Gamma_f$.

**Solution:** Define $g : X \to \Gamma_f$ by $g(x) = (x, f(x))$. Since $X \times Y$ has the product topology and $g$ is continuous in each coordinate, we get that $g$ is continuous.

Define $h : \Gamma_f \to X$ by $h(x, y) = x$. Since $h$ is the restriction of the standard projection $X \times Y \to X$ to $\Gamma_f$, we get that $h$ is continuous.

For $x \in X$ we have

$$h(g(x)) = h(x, f(x)) = x.$$ 

For $(x, f(x)) \in \Gamma_f$ we have

$$g(h(x, f(x))) = g(x) = (x, f(x)).$$

We conclude that $g$ and $h$ are mutually inverse continuous maps, and hence mutually inverse homeomorphisms. Therefore, we get that $X$ is homeomorphic to $\Gamma_f$, as desired.

**Problem 10:** (Optional) This problem is about continuous maps in the context of the Zariski topology introduced on the optional problem of the last homework.
A Zariski closed subset $V \subseteq \mathbb{R}^n$ is called a variety. Recall that this means that $V$ is the set of common zeros $V(E)$ of some collection $E$ of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$.

(1) Give some examples of varieties that you have seen before in high school. Prove that the set of singular $n \times n$ real matrices is a variety.

Suppose $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ are varieties. A polynomial map $\psi : V \to W$ is a function given by a list $\psi = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ of $m$ polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ with the property that, for all $(a_1, \ldots, a_n) \in V$ we have $\psi(a_1, \ldots, a_n) = (f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)) \in W$.

(2) Suppose that $V = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 1 = 0\}$ is the unit circle. Give two different polynomials $f(x, y), g(x, y) \in \mathbb{R}[x, y]$ such that the polynomial maps $S^1 \to \mathbb{R}$ induced by $f$ and $g$ coincide.

(3) Suppose $\psi : V \to W$ is a polynomial map between varieties $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$. Prove that $\psi$ is continuous in the Zariski topology (i.e., the subspace topologies on $V$ and $W$ inherited from the Zariski topologies on $\mathbb{R}^n$ and $\mathbb{R}^m$).