Problem 1: Let $J$ be a nonempty (possibly infinite) set and endow $\mathbb{R}$ with the standard topology. Endow the product $\mathbb{R}^J$ with the product topology. Let $f_n : J \to \mathbb{R}$ be a sequence in $\mathbb{R}^J$. Let $f : J \to \mathbb{R}$ be another element of $\mathbb{R}^J$. Prove that the following two conditions are equivalent.

1. The sequence $f_n$ converges to $f$ in $\mathbb{R}^J$ in the product topology.
2. For any $j \in J$, we have that $f_n(j) \to f(j)$ in $\mathbb{R}$ as $n \to \infty$.

For this reason, the product topology on $\mathbb{R}^J$ is sometimes called the “topology of pointwise convergence”.

Problem 2: Let $J$ be a nonempty (possibly infinite) set and endow $\mathbb{R}$ with the standard topology. Endow the product $\mathbb{R}^J$ with the box topology. Let $f_n : J \to \mathbb{R}$ be a sequence in $\mathbb{R}^J$. Let $f : J \to \mathbb{R}$ be another element of $\mathbb{R}^J$. Consider the following two conditions.

1. The sequence $f_n$ converges to $f$ in $\mathbb{R}^J$ in the box topology.
2. For any $\epsilon > 0$, there exists $N$ such that $n \geq N$ implies $|f_n(j) - f(j)| < \epsilon$ for any $j \in J$.

Prove that (1) $\Rightarrow$ (2) but that the converse does not necessarily hold. (So that the box topology does not deserve the name “the topology of uniform convergence”.)

Optional: And can you characterize convergence in the box topology?

Problem 3: Consider the set $\mathbb{R}^\omega := \mathbb{R} \times \mathbb{R} \times \cdots$ of all sequences $(a_1, a_2, \ldots)$ of real numbers. Let $\mathbb{R}^\infty \subset \mathbb{R}^\omega$ be the collection

$$\mathbb{R}^\infty := \{(a_1, a_2, \ldots) : \text{there exists } N \text{ such that } a_n = 0 \text{ for } n \geq N\}$$

of real sequences which are eventually zero. Calculate the closure $\overline{\mathbb{R}^\infty}$ of $\mathbb{R}^\infty$ in both the product and box topologies.

Problem 4: Let $S_\Omega$ be the minimal uncountable well ordered set and endow the ordered set $\overline{S_\Omega} = S_\Omega \cup \{\Omega\}$ with the order topology. Prove that $\overline{S_\Omega}$ is not metrizable (and so the subspace $S_\Omega$ is not metrizable). This gives an example of a Hausdorff space which is not metrizable.

Problem 5: Let $(X, d)$ be a metric space. Prove that the function $d : X \times X \to \mathbb{R}_{\geq 0}$ is continuous.

Problem 6: Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The disjoint union $\coprod_{\alpha \in J} X_\alpha$ topology has underlying set given by the disjoint union of the $X_\alpha$ and open sets $\coprod_{\alpha \in J} U_\alpha$, where $U_\alpha \subset X_\alpha$ is open for all $\alpha \in J$.

1. For any $\alpha_0 \in J$, prove that the inclusion mapping

$$i_{\alpha_0} : X_{\alpha_0} \hookrightarrow \coprod_{\alpha \in J} X_\alpha$$

is continuous.
(2) Let $B$ be another topological space and let \( \{ f_\alpha : X_\alpha \to B \}_{\alpha \in J} \) be a family of continuous maps. Prove that there exists a unique continuous map $f : \Pi_{\alpha \in J} X_\alpha \to B$ such that
\[
 f_\alpha = f \circ \iota_\alpha
\]
for all $\alpha \in J$.

Observe that Part 2 above is similar to the universal property satisfied by the product topology, but the maps go in the opposite direction! In jargon, we say that $\Pi_{\alpha \in J} X_\alpha$ is the \textit{coproduct} of the $X_\alpha$ in the category of topological spaces.

**Problem 7:** (Optional.) A \textit{group} $G$ is a set with a binary operation $\cdot : G \times G \to G$ such that

- (associativity) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in G$.
- (identity) There exists $e \in G$ such that $x \cdot e = e \cdot x = x$ for all $x \in G$.
- (inverse) For any $x \in G$, there exists $y \in G$ such that $x \cdot y = y \cdot x = e$.

The group element $y$ in the last axiom is determined by $x$; we write $y = x^{-1}$.

A \textit{topological group} is a group $G$ which is also a topological space such that singleton sets are closed (i.e., $G$ satisfies the $T_1$ axiom) and the maps $m : G \times G \to G$ and $i : G \to G$ given by $m(x, y) = x \cdot y$ and $i(x) = x^{-1}$ are continuous.

1. Explain how the following topological spaces are topological groups.
   (a) The real line $\mathbb{R}$ in the standard topology. More generally, Euclidean space $\mathbb{R}^n$.
   (b) The group $GL(n, \mathbb{R})$ of invertible $n \times n$ real matrices, under matrix multiplication.
   (c) The circle $S^1$. More generally, the $n$-dimensional torus $(S^1)^n$.

2. Let $G$ be a group which is also a topological space satisfying the $T_1$ axiom. Prove that $G$ is a topological group if and only if the map $f : G \times G \to G$ given by $f(x, y) = x \cdot y^{-1}$ is continuous.

3. A \textit{subgroup} of a group $G$ is a subset $H \subseteq G$ such that $e \in G$ and $H$ is closed under multiplication and inversion. If $G$ is a topological group and $H$ is a subgroup of $G$, prove that both $H$ and $\overline{H}$ are topological groups.

4. Let $G$ be a topological group and let $g \in G$. Define functions $r_g : G \to G$ and $\ell_g : G \to G$ by $r_g(x) = x \cdot g$ and $\ell_g(x) = g \cdot x$. (These are the so-called right and left translations by $g$.) Prove that both $r_g$ and $\ell_g$ are homeomorphisms of $G$ onto itself. (Observe that $r_g(e) = \ell_g(e) = g$. This says that $G$ “looks the same” close to $e$ as close to $g$. This is expressed by saying that $G$ is a “homogeneous space”.)

5. What should the Category of Topological Groups be? (What should the morphisms look like?)

It can be shown that topological groups are Hausdorff. In fact, topological groups satisfy the \textit{regularity axiom}:

A topological space $X$ is \textit{regular} if singleton sets are closed and, for every point $x \in X$ and any closed set $A \subset X$ with $x \notin A$, there exist open sets $U$ and $V$ in $X$ such that $x \in U$, $A \subset V$, and $U \cap V = \emptyset$. 
If the group $G$ is not merely a topological space, but also a smooth manifold (and the multiplication and inversion maps are not merely continuous, but smooth), then $G$ is called a Lie group. All of the examples given in Part 1 of this problem are actually Lie groups.