Problem 1: Give $\mathbb{R}$ the standard topology and let $\mathbb{R}/\mathbb{Q}$ be the quotient space obtained by identifying $\mathbb{Q} \subset \mathbb{R}$ to a point. Prove that $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

Problem 2: Let $D^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$ denote the closed $n$-dimensional disc and let $S^{n-1}$ be its boundary. Let $D^n/S^{n-1}$ be the quotient space obtained by identifying $S^{n-1}$ to a point. Exhibit a continuous bijection $f : D^n/S^{n-1} \rightarrow S^n$.

(Remark: Since $D^n/S^{n-1}$ is compact and $S^n$ is Hausdorff, any continuous bijection $D^n/S^{n-1} \rightarrow S^n$ is automatically a homeomorphism.)

Problem 3: Let $\mathbb{R}_1$ and $\mathbb{R}_2$ be two copies of the real line $\mathbb{R}$ in the standard topology and consider the disjoint union topology on $\mathbb{R}_1 \sqcup \mathbb{R}_2$. Define an equivalence relation $\sim$ on $\mathbb{R}_1 \sqcup \mathbb{R}_2$ by setting $x_1 \sim x_2$ for all $x \neq 0$, where $x_i$ is the copy of $x$ in $\mathbb{R}_i$ for $i = 1, 2$. Let $X := (\mathbb{R}_1 \sqcup \mathbb{R}_2)/\sim$ be the resulting quotient space. $X$ is called the “line with two origins”.

Prove that $X$ is not Hausdorff.

Problem 4: Let $X$ be the line with two origins from the last problem. Prove that, for every points $x \in X$, there is a neighborhood $U$ of $x$ which is homeomorphic to Euclidean space.

Problem 5:

(1) Prove that no two of $(0,1), [0,1)$, and $[0,1]$ are homeomorphic.

(2) Prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$ for $n > 1$.

In fact, $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $n = m$, but this is harder to prove.

Problem 6: Prove that an infinite set $X$ is connected in the finite complement topology.

Problem 7: A space $X$ is totally disconnected if its only connected components are singletons. Prove that discrete spaces are totally disconnected, but that the converse does not hold.

Problem 8: (Optional; requires some algebra.) Let $R$ be a commutative ring with a $1 \neq 0$. Recall that a prime ideal $P \subset R$ is a proper ideal such that $xy \in P$ implies $x \in P$ or $y \in P$ for any $x, y \in R$. The prime spectrum of $R$ is

$$\text{Spec}(R) := \{\text{all prime ideals } P \subset R\}.$$ 

This problem studies $\text{Spec}(R)$ as a topological space.

(1) For any subset $E \subseteq R$, let $V(E) \subseteq \text{Spec}(R)$ be given by

$$V(E) := \{P \in \text{Spec}(R) : E \subseteq P\}.$$
Prove that the sets \( \{ \mathbf{V}(E) : E \subseteq R \} \) define the closed sets for a topology on \( \text{Spec}(R) \). This is called the **Zariski topology** on \( \text{Spec}(R) \).

(2) Let \( R = \mathbb{Z} \) be the ring of integers. Describe \( \text{Spec}(\mathbb{Z}) \) (i.e., what are the points? What about the closed sets?)

(3) For any ring \( R \), the **maximal spectrum** of \( R \) is the subspace \( \text{mSpec}(R) \subseteq \text{Spec}(R) \) given by
\[
\text{mSpec}(R) := \{ M \subseteq R : M \text{ a maximal ideal} \}.
\]

Let \( R = \mathbb{C}[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables with coefficients in \( \mathbb{C} \). Explain why \( \text{mSpec}(R) \) is ‘the same’ as the Zariski topology on \( \mathbb{C}^n \) explained in a previous optional problem. (Hint: **Hilbert’s Nullstellensatz** says that the maximal ideals in \( \mathbb{C}[x_1, \ldots, x_n] \) are precisely those of the form \( \langle x_1 - a_1, \ldots, x_n - a_n \rangle \), for \( (a_1, \ldots, a_n) \in \mathbb{C}^n \). This gives a bijection between \( \text{mSpec}(R) \) and \( \mathbb{C}^n \).

(4) Let \( R \) and \( S \) be rings. Prove that \( \text{Spec}(R \oplus S) \) is homeomorphic to the disjoint union \( \text{Spec}(R) \amalg \text{Spec}(S) \).

(5) Let \( \varphi : R \rightarrow S \) be a homomorphism of rings sending \( 1_R \) to \( 1_S \). Explain how \( \varphi \) induces a continuous map
\[
\varphi^\# : \text{Spec}(S) \longrightarrow \text{Spec}(R).
\]

(Note the direction of the arrow! In jargon, \( \text{Spec}(\cdot) \) is a contravariant functor from the category of rings to the category of topological spaces. In fact, the space \( \text{Spec}(R) \) can be augmented with a ‘structure sheaf’ \( O_{\text{Spec}(R)} \) to give it the structure of a scheme. Then \( \text{Spec}(\cdot) \) is a contravariant functor from the category of rings to the category of schemes.)