Problem 1: Prove that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to $S^n$.

Solution: We know that $\mathbb{R}^n$ is a locally compact Hausdorff space (which is not compact) and that $S^n$ is a compact Hausdorff space. By the uniqueness of the one-point compactification, it is enough to show that there is a subspace $X$ of $S^n$ such that $S^n - X$ is a single point and $X$ is homeomorphic to $\mathbb{R}^n$. But we know that $X = S^n - \{N\}$, where $N = (1,0,\ldots,0)$ is such a subspace (where the homeomorphism with $\mathbb{R}^n$ is afforded by stereographic projection).

Problem 2: Prove that the one-point compactification of $S_\Omega$ is homeomorphic to $S_\Omega$.

Solution: Both of the spaces $S_\Omega$ and $\overline{S_\Omega}$ are order topologies, and hence Hausdorff. The sets $S_\Omega$ and $\overline{S_\Omega}$ are both well orders. In particular, they have the least upper bound property. If $a_0 \in S_\Omega$ is the smallest element, we have that $S_\Omega = [a_0, \Omega]$ is compact. If $x \in S_\Omega$ and $x < y < \Omega$, we have that $x \in [a_0, y) \subset [a_0, y]$ with $[a_0, y]$ compact, so that $S_\Omega$ is locally compact. Finally, we have that $\overline{S_\Omega} - S_\Omega = \{\Omega\}$ is a single point. By the uniqueness of the one-point compactification of a locally compact Hausdorff space, it follows that the one-point compactification of $S_\Omega$ is homeomorphic to $\overline{S_\Omega}$.

Problem 3: Let $(X,d)$ be a metric space and assume that $X$ has a countable dense subset (i.e., $X$ is separable). Prove that $X$ is second-countable.

Solution: Let $D = \{x_1, x_2, \ldots\}$ be a countable dense subset of $X$ and let $B := \{B_d(x_n, 1/m) : n, m \in \mathbb{Z}_{>0}\}$.

Then $B$ is a countable collection of open sets in $X$.

We claim that $B$ is a basis for the topology of $X$. To see this, let $U \subset X$ be open and let $y \in U$. It is enough to show that there exists $B \in B$ such that $y \in B \subset U$. Indeed, let $0 < \epsilon < 1$ be such that $B_d(y, \epsilon) \subset U$. Since $D$ is dense, there exists $n \in \mathbb{Z}_{>0}$ such that $x_n \in B_d(y, \epsilon/4)$. Choose $m \in \mathbb{Z}_{>0}$ so that $\epsilon/4 < 1/m < \epsilon/2$. Since $\epsilon/4 < 1/m$, we have that $y \in B_d(x_n, 1/m)$. On the other hand, if $z \in B_d(x_n, 1/m)$, we have that $d(y, z) \leq d(y, x_n) + d(x_n, z) < \epsilon/4 + 1/m < 3\epsilon/4 < \epsilon$. It follows that $y \in B_d(x_n, 1/m) \subset B_d(y, \epsilon) \subset U$, so that $B$ is a countable basis for the topology on $X$ and $X$ is second-countable.

Problem 4: Suppose $X$ is a second-countable space and $A \subset X$ is an uncountable subset. Prove that uncountably many points in $A$ are limit points of $A$.

Solution: Let $B = \{B_1, B_2, \ldots\}$ be a countable basis for the topology on $X$. We define a function $\varphi : (A - A') \rightarrow \mathbb{Z}_{>0}$.
as follows. For any \( a \in (A - A') \), there exists a basic open set \( B_{n_a} \) such that \( A \cap B_{n_a} = \{a\} \). Choose one such \( n_a \) and set \( \varphi(a) := n_a \). We claim that \( \varphi \) is an injection. Indeed, for \( a_1 \neq a_2 \) in \( A - A' \), we have that \( A \cap B_{\varphi(a_1)} = \{a\} \neq \{a'\} = A \cap B_{\varphi(a')} \). The injectivity of \( \varphi \) implies that \( A - A' \) is countable. Since \( A \) is uncountable, this forces \( A' \) to be uncountable.

**Problem 5:** Which of the four countability axioms (first-countable, second-countable, Lindelöf, separable) does \( S_\Omega \) satisfy? What about \( \overline{S_\Omega} \)?

**Solution:** We claim that \( S_\Omega \) is not separable. Indeed, if \( A \subset S_\Omega \) is countable, then there is an upper bound \( b \) for \( A \). Choosing \( b' > b \), we get that \( (b, \infty) \) is a neighborhood of \( b' \) which does not meet \( A \). Thus, \( b' \notin \overline{A} \). It follows that \( S_\Omega \) is also not second-countable.

\( S_\Omega \) is first-countable. Indeed, for any \( x \in S_\Omega \), let \( x' \) be the immediate successor of \( x \). Provided \( x \neq a_0 \), we have that \( \{(z, x') : z < x\} \) is a countable basis at \( x \). If \( x = a_0 \), then \( \{(x, x')\} \) is a singleton basis at \( x \).

We claim that \( \overline{S_\Omega} \) is not first-countable. Indeed, there does not exist a countable basis at \( \Omega \). If \( \{B_1, B_2, \ldots\} \) were such a basis, for all \( n \geq 0 \) there would be \( x_n \in S_\Omega \) such that \( (x_n, \Omega) \subset B_n \). Let \( b \) be an upper bound in \( S_\Omega \) of the countable set \( \{x_1, x_2, \ldots\} \) and let \( b' \) be the immediate successor of \( b \). Then \( (b', \Omega) \) is a neighborhood of \( \Omega \) containing none of the \( B_n \). This completes the proof that \( \overline{S_\Omega} \) is not first-countable. It follows that \( \overline{S_\Omega} \) also fails to be second-countable.

\( \overline{S_\Omega} \) is not separable. If \( A \subset \overline{S_\Omega} \) is countable, let \( b \in S_\Omega \) be an upper bound for \( A \cap S_\Omega \). If \( b' \) is the immediate successor of \( b \), we get that \( (b, \Omega) \) is a neighborhood of \( b' \) which does not meet \( A \). Therefore, \( b' \notin \overline{A} \).

\( \overline{S_\Omega} \) is Lindelöf. To see this, let \( U \) be an open cover of \( \overline{S_\Omega} \). There exists \( U \in \mathcal{U} \) such that \( \Omega \in U \). Choose \( x \in S_\Omega \) such that \( (x, \Omega) \subset U \). We have that \( [a_0, x) \) is countable. Write \( [a_0, x) = \{y_1, y_2, \ldots\} \). For every \( n \), there exists \( U_n \in \mathcal{U} \) such that \( y_n \in U_n \). Now \( \{U, U_1, U_2, \ldots\} \) is a countable subcover of \( \mathcal{U} \).

**Problem 6:** Let \( X \) be a regular space and let \( x, y \in X \) be distinct points. Prove that there exist neighborhoods \( U, V \) of \( x, y \) such that \( \overline{U} \cap \overline{V} = \emptyset \).

**Solution:** Since \( X \) is regular, \( X \) is Hausdorff. Choose neighborhoods \( U \) of \( x \) and \( V \) of \( y \) such that \( U \cap V = \emptyset \). Now \( X - W \) is a closed set with \( x \notin X - W \). By regularity, there is a neighborhood \( V \) of \( y \) and an open set \( V' \) containing \( X - W \) such that \( V \cap V' = \emptyset \). Now \( X - V' \) is a closed set with \( \overline{V} \subset X - V' \). In particular, we get that \( \overline{V} \subset X - V' \). On the other hand, we have \( \overline{U} \subset X - W \). But \( (X - V') \cap (X - W) = \emptyset \), forcing \( \overline{U} \cap \overline{V} = \emptyset \).

**Problem 7:** Prove that every order topology is regular.

**Solution:** Let \( X \) be an order topology, let \( x \in X \), and let \( A \subset X \) be a closed set with \( x \notin A \).
Suppose $x$ is the smallest element of $X$. If $x'$ is the immediate successor of $x$, then 
$[x, x')$ and $(x, \infty)$ are disjoint neighborhoods of $x$ and $A$, respectively. If $x$ has no
immediate successor, choose $y > x$ such that $[x, y) \cap A = \emptyset$. Now choose $y' \in (x, y)$.
We have that $[x, y')$ and $(y'\infty)$ are disjoint neighborhoods of $x$ and $A$, respectively.

If $x$ is the largest element of $X$, we reason as in the previous paragraph, with the
order $<$ reversed.

If $x$ is neither smallest nor largest, consider the closed subspaces $X_1 = (-\infty, x]$ and
$X_2 = [x, \infty)$ of $X$ and let $A_i = X \cap X_i$ for $i = 1, 2$. Then $A_i$ is closed in $X_i$ and,
by the previous arguments, there exist neighborhoods $U_i$ of $A_i$ and $V_i$ of $x$ in $X_i$ such
that $U_i \cap V_i = \emptyset$. We get that $U \cap V = \emptyset$, where $U = U_1 \cup U_2$ and $V = V_1 \cup V_2$. We
claim that $U$ and $V$ are open in $X$. Indeed, we have that $V_1$ is open in the open ray
$(-\infty, x)$ and $V_2$ is open in the open ray $(x, \infty)$. Similarly, $U$ contains an open interval
containing $x$ and $U_i - \{x\}$ is open in $X$ for $i = 1, 2$. 