Problem 1: Recall that an \( n \times n \) real matrix \( A \) is called \emph{orthogonal} if \( AA^T = I_n \), where \( A^T \) is the transpose of \( A \) and \( I_n \) is the \( n \times n \) identity matrix. Prove that the orthogonal group
\[
O(n) := \{ A \in \text{Mat}_n(\mathbb{R}) : A \text{ is orthogonal} \}
\]
forms a closed subset of matrix space \( \text{Mat}_n(\mathbb{R}) \).

Problem 2: Let \( f : X \to Y \) be a continuous function between topological spaces. The \emph{graph} of \( f \) is the subspace \( \Gamma_f \subset X \times Y \) given by
\[
\Gamma_f := \{ (x, f(x)) : x \in X \}.
\]
Prove that \( X \) is homeomorphic to \( \Gamma_f \).

Problem 3: Let \( Y \) be an ordered set in the order topology and suppose that \( f, g : X \to Y \) are continuous functions (where \( X \) is some topological space).

1. Prove that \( \{ x \in X : f(x) \leq g(x) \} \) is closed in \( X \).
2. Prove that the function \( h : X \to Y \) given by \( h(x) := \min(f(x), g(x)) \) is continuous. (Hint: Pasting Lemma)

Problem 4: Let \( \mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots \) be the set of all infinite sequences \( (a_1, a_2, \ldots) \) of real numbers and let \( \mathbb{R}^\infty \subset \mathbb{R}^\omega \) be the set of sequences \( (a_1, a_2, \ldots) \) that are \emph{eventually zero} (i.e., there exists \( N \) such that \( n \geq N \) implies \( a_n = 0 \)). Calculate the closure of \( \mathbb{R}^\infty \) inside \( \mathbb{R}^\omega \) if \( \mathbb{R}^\omega \) is given

1. the product topology, and
2. the box topology.

Problem 5: Look up “stereographic projection”. Let \( S^n \) be the \( n \)-sphere and let \( N := (1, 0, 0, \ldots, 0) \) be the ‘north pole’. Prove that \( \mathbb{R}^n \) is homeomorphic to \( S^n - \{ N \} \).

Problem 6: A subset \( D \) of a space \( X \) is called \emph{dense} if \( \overline{D} = X \). Let \( X \) and \( Y \) be spaces with \( Y \) Hausdorff. Let \( D \subset X \) be a dense subset. Let \( f, g : X \to Y \) be continuous functions such that \( f(x) = g(x) \) for all \( x \in D \). Prove that \( f = g \). Does this necessarily hold if \( Y \) is not Hausdorff?

Problem 7: Let \( J \) be a (possibly uncountable) set and endow \( \mathbb{R}^J \) with the product topology. Let \( f_n : J \to \mathbb{R} \) be a sequence in \( \mathbb{R}^J \) and let \( f : J \to \mathbb{R} \) be a point in \( \mathbb{R}^J \).
Prove that the following two conditions are equivalent:

1. We have \( f_n \to f \) in the product topology on \( \mathbb{R}^J \).
2. For each \( j \in J \), we have \( f_n(j) \to f(j) \) in \( \mathbb{R} \).
For this reason, the product topology is sometimes called “the topology of pointwise convergence”.

Problem 8: (Optional - not to be handed in.) A topological group is a group $G$ which is also a topological space satisfying the $T_1$ axiom (i.e. singleton sets are closed) such that the maps $m : G \times G \to G$ and $i : G \to G$ given by $m(x,y) = x \cdot y$ and $i(x) = x^{-1}$ are continuous.

(1) Explain how the following objects are topological groups.
   (a) Euclidean space $\mathbb{R}^n$.
   (b) The set $GL_n(\mathbb{R})$ of invertible $n \times n$ real matrices.
   (c) The circle $S^1$.

(2) Let $G$ be a topological group. Prove that the map $f : G \times G \to G$ given by $f(x,y) = x \cdot y^{-1}$ is continuous.

(3) Let $G$ be a topological group and let $H$ be a subgroup of $G$. Prove that both $H$ and $\overline{H}$ (i.e. the closure of $H$ within $G$) are topological groups.

(4) Let $G$ be a topological group. For a fixed $g \in G$, define a map $r_g : G \to G$ by $r_g(x) = g \cdot x$. Prove that $r_g$ is a homeomorphism. (This means that $G$ “looks the same” near any point. That is, $G$ is a “homogeneous” space.)

(5) What should a morphism in the Category of Topological Groups be?