Math 190: Winter 2018
Homework 3
Due 5:00pm on Thursday 2/1/2018

Problem 1: Recall that an $n \times n$ real matrix A is called *orthogonal* if $AA^T = I_n$, where A^T is the transpose of A and I_n is the $n \times n$ identity matrix. Prove that the *orthogonal group*

$$O(n) := \{A \in \mathrm{Mat}_n(\mathbb{R}) \,:\, A \text{ is orthogonal}\}$$

forms a closed subset of matrix space $\mathrm{Mat}_n(\mathbb{R})$.

Problem 2: Let $f : X \to Y$ be a continuous function between topological spaces. The *graph* of f is the subspace $\Gamma_f \subset X \times Y$ given by

$$\Gamma_f := \{(x, f(x)) \,:\, x \in X\}.$$

Prove that X is homeomorphic to Γ_f.

Problem 3: Let Y be an ordered set in the order topology and suppose that $f, g : X \to Y$ are continuous functions (where X is some topological space).

(1) Prove that $\{x \in X \,:\, f(x) \leq g(x)\}$ is closed in X.
(2) Prove that the function $h : X \to Y$ given by $h(x) := \min(f(x), g(x))$ is continuous. (Hint: Pasting Lemma)

Problem 4: Let $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots$ be the set of all infinite sequences $(a_1, a_2, \dots)$ of real numbers and let $\mathbb{R}^\infty \subset \mathbb{R}^\omega$ be the set of sequences $(a_1, a_2, \dots)$ that are *eventually zero* (i.e., there exists N such that $n \geq N$ implies $a_n = 0$). Calculate the closure of $\mathbb{R}^\infty$ inside $\mathbb{R}^\omega$ if $\mathbb{R}^\omega$ is given

(1) the product topology, and
(2) the box topology.

Problem 5: Look up "stereographic projection". Let S^n be the n-sphere and let

$$N := (1, 0, 0, \dots, 0)$$

be the 'north pole'. Prove that $\mathbb{R}^n$ is homeomorphic to $S^n - \{N\}$.

Problem 6: A subset D of a space X is called *dense* if $\overline{D} = X$. Let X and Y be spaces with Y Hausdorff. Let $D \subset X$ be a dense subset. Let $f, g : X \to Y$ be continuous functions such that $f(x) = g(x)$ for all $x \in D$. Prove that $f = g$. Does this necessarily hold if Y is not Hausdorff?

Problem 7: Let J be a (possibly uncountable) set and endow $\mathbb{R}^J$ with the product topology. Let $f_n : J \to \mathbb{R}$ be a sequence in $\mathbb{R}^J$ and let $f : J \to \mathbb{R}$ be a point in $\mathbb{R}^J$. Prove that the following two conditions are equivalent:

(1) We have $f_n \to f$ in the product topology on $\mathbb{R}^J$.
(2) For each $j \in J$, we have $f_n(j) \to f(j)$ in $\mathbb{R}$.

For this reason, the product topology is sometimes called "the topology of pointwise convergence".

Problem 8: (Optional - not to be handed in.) A *topological group* is a group G which is also a topological space satisfying the T_1 axiom (i.e. singleton sets are closed) such that the maps $m : G \times G \to G$ and $i : G \to G$ given by $m(x, y) = x \cdot y$ and $i(x) = x^{-1}$ are continuous.

(1) Explain how the following objects are topological groups.
   (a) Euclidean space $\mathbb{R}^n$.
   (b) The set $GL_n(\mathbb{R})$ of invertible $n \times n$ real matrices.
   (c) The circle S^1.
(2) Let G be a topological group. Prove that the map $f : G \times G \to G$ given by $f(x, y) = x \cdot y^{-1}$ is continuous.
(3) Let G be a topological group and let H be a subgroup of G. Prove that both H and $\overline{H}$ (i.e. the closure of H within G) are topological groups.
(4) Let G be a topological group. For a fixed $g \in G$, define a map $r_g : G \to G$ by $r_g(x) = g \cdot x$. Prove that r_g is a homeomorphism. (This means that G "looks the same" near any point. That is, G is a "homogeneous" space.)
(5) What should a morphism in the Category of Topological Groups be?