Problem 1: Let \((X,d)\) be a metric space. Prove that the function \(d : X \times X \to \mathbb{R}_{\geq 0}\) is continuous.

Problem 2: Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. Let \(f : X \to Y\) be a function satisfying \(d_X(x_1,x_2) = d_Y(f(x_1),f(x_2))\) for all \(x_1,x_2 \in X\). Prove that \(f\) is an imbedding.

Problem 3: Let \(\{X_\alpha\}_{\alpha \in J}\) be a family of topological spaces and consider the disjoint union \(\bigsqcup_{\alpha \in J} X_\alpha\) of the sets \(X_\alpha\). The disjoint union topology on \(\bigsqcup_{\alpha \in J} X_\alpha\) has open sets \(\bigsqcup_{\alpha \in J} U_\alpha\), where \(U_\alpha \subset X_\alpha\) is open for all \(\alpha \in J\).

- For each \(\alpha_0 \in J\), let \(\iota_{\alpha_0} : X_{\alpha_0} \hookrightarrow \bigsqcup_{\alpha \in J} X_\alpha\) be the inclusion. Prove that \(\iota_{\alpha_0}\) is continuous.
- Prove the following universal property:
  If \(A\) is a topological space and \(f_\alpha : X_\alpha \rightarrow A\) is a family of continuous maps (for all \(\alpha \in J\)), there exists a unique continuous map \(f : \bigsqcup_{\alpha \in J} X_\alpha \rightarrow A\) such that \(f_\alpha = f \circ \iota_\alpha\) for all \(\alpha \in J\).
You have proven that \(\bigsqcup_{\alpha \in J} X_\alpha\) serves as the coproduct in the category of topological spaces.

Problem 4: Give \(\mathbb{R}\) the standard topology and let \(\mathbb{R}/\mathbb{Q}\) be the quotient space obtained by identifying \(\mathbb{Q}\) to a point. Prove that \(\mathbb{R}/\mathbb{Q}\) is not Hausdorff.

Problem 5: Consider the disjoint union \(\mathbb{R}_1 \bigsqcup \mathbb{R}_2\) of two copies of the real line. Let \(\sim\) be the equivalence relation on \(\mathbb{R}_1 \bigsqcup \mathbb{R}_2\) given by setting \(x_1 \sim x_2\) if \(x_i\) is the copy of \(x\) in \(\mathbb{R}_i\) for \(i = 1, 2\) and \(x \neq 0\). Let \(X := \mathbb{R}_1 \bigsqcup \mathbb{R}_2/ \sim\) be the associated quotient space. Prove that \(X\) is not Hausdorff.

\(X\) is called the ‘line with two origins’.

Problem 6: Let \(X = \mathbb{R}^{n+1} - \{0\}\) be \((n+1)\)-dimensional Euclidean space with the origin removed. Define an equivalence relation \(\sim\) on \(X\) by the rule \(v \sim \lambda v\) for all vectors \(v \in X\) and scalars \(\lambda \neq 0\). The \(n\)-dimensional real projective space is the quotient \(\mathbb{P}^n(\mathbb{R}) := X/\sim\). Prove that \(\mathbb{P}^n(\mathbb{R})\) is Hausdorff.

We can think about \(\mathbb{P}^n(\mathbb{R})\) as the set of lines through the origin in \(\mathbb{R}^{n+1}\). We can define complex projective space \(\mathbb{P}^n(\mathbb{C})\) in the same way; it is also Hausdorff.

Problem 7: Let \(D^n\) be the closed \(n\)-dimensional unit disc

\[D^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}\]

and let \(S^{n-1}\) be its boundary. Exhibit a continuous bijection \(f : D^n/S^{n-1} \rightarrow S^n\). (We will soon have the technology to prove that any such continuous bijection \(f\) must be a homeomorphism – \(S^n\) is Hausdorff and \(D^n/S^{n-1}\) is compact.)
Problem 8: Let $M$ be the Möbius strip and let $D = D^2$ be the closed 2-dimensional disc. Prove that the quotient space $X = (M \amalg D) / \sim$ obtained by gluing the boundary circle of $M$ to the boundary circle of $D$ is homeomorphic to 2-dimensional real projective space $\mathbb{P}^2(\mathbb{R})$. (Hint: Model $\mathbb{P}^2(\mathbb{R})$ as the sphere $S^2$ with antipodal points identified. Constructing the actual maps here is tedious; intuitive ‘cut and paste arguments’ are OK. Consider the ‘arctic and antarctic circles’ on $S^2$.)

Problem 9: (Optional - not to be handed in.) Let $k \leq n$ be positive integers. The (real) Grassmannian $Gr(k, n)$ is the family of all $k$-dimensional subspaces of $\mathbb{R}^n$:

$$Gr(k, n) = \{ V \text{ a subspace of } \mathbb{R}^n : \dim(V) = k \}.$$ 

(a) Explain why $Gr(1, n)$ is the same as projective space $\mathbb{P}^{n-1}(\mathbb{R})$. We give $Gr(k, n)$ a topology as follows. Let $U_{k,n} \subset \text{Mat}_{k \times n}(\mathbb{R})$ be the family of full rank $k \times n$ real matrices.

(b) Prove that $U_{k,n}$ is an open subset of $\text{Mat}_{k \times n}(\mathbb{R})$. (Hint: Having full rank means something about the minors, i.e. sub-determinants, of $M$.) The group $GL_k(\mathbb{R})$ acts on $U_{k,n}$ by the rule $A.M = AM$ for all $A \in GL_k(\mathbb{R})$ and $M \in U_{k,n}$.

(c) Prove that $Gr(k, n)$ may be identified with the set of orbits $GL_k(\mathbb{R}) \backslash U_{k,n} := \{ GL_k(\mathbb{R}).M : M \in U_{k,n} \}$. (Hint: Linear algebra.) We may now endow $Gr(k, n)$ with the quotient topology.

A partition is a way to shove boxes into a corner. We will consider shoving $1 \times 1$ boxes into the northeast corner of a $k \times (n - k)$ room (this corner choice is called Japanese notation). If we have a matrix $M \in U_{n,k}$, there is a unique matrix $A \in GL_k(\mathbb{R})$ such that $AM$ is in row reduced echelon form (why?). If $n = 9$ and $k = 4$, such a RREF matrix might look like this:

$$\begin{pmatrix}
0 & 1 & * & * & 0 & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 1 & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 1 & * & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & *
\end{pmatrix}$$

As you can see, the stars in the above picture form a (Japanese) partition:

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We might call this partition $(4, 2, 2, 1)$. The open Schubert cell $C_\lambda \subset Gr(k, n)$ corresponding to a partition $\lambda \subset k \times (n - k)$ is the family of all orbits which can be represented by matrices with RREF corresponding to $\lambda$.

(d) Prove that we have a disjoint union decomposition $Gr(k, n) = \bigsqcup_{\lambda \subset k \times (n-k)} C_\lambda$.

(e) Prove that $C_\lambda$ is homeomorphic to $\mathbb{R}^{|\lambda|}$, where $|\lambda|$ is the number of boxes in $\lambda$. (This requires ingenuity – how can you write down your homeomorphism?)