Math 190: Winter 2018
Homework 4
Due 5:00pm on Thursday 2/15/2018

Problem 1: Let (X, d) be a metric space. Prove that the function $d : X \times X \to \mathbb{R}_{\geq 0}$ is continuous.

Problem 2: Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a function satisfying $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. Prove that f is an imbedding.

Problem 3: Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and consider the disjoint union $\amalg_{\alpha \in J} X_\alpha$ of the sets X_α. The *disjoint union topology* on $\amalg_{\alpha \in J} X_\alpha$ has open sets $\amalg_{\alpha \in J} U_\alpha$, where $U_\alpha \subset X_\alpha$ is open for all $\alpha \in J$.

- For each $\alpha_0 \in J$, let $\iota_{\alpha_0} : X_{\alpha_0} \hookrightarrow \amalg_{\alpha \in J} X_\alpha$ be the inclusion. Prove that ι_{α_0} is continuous.
- Prove the following universal property:

  > If A is a topological space and $f_\alpha : X_\alpha \to A$ is a family of continuous maps (for all $\alpha \in J$), there exists a unique continuous map $f : \amalg_{\alpha \in J} X_\alpha \to A$ such that $f_\alpha = f \circ \iota_\alpha$ for all $\alpha \in J$.

  You have proven that $\amalg_{\alpha \in J} X_\alpha$ serves as the *coproduct* in the category of topological spaces.

Problem 4: Give $\mathbb{R}$ the standard topology and let $\mathbb{R}/\mathbb{Q}$ be the quotient space obtained by identifying $\mathbb{Q}$ to a point. Prove that $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

Problem 5: Consider the disjoint union $\mathbb{R}_1 \amalg \mathbb{R}_2$ of two copies of the real line. Let $\sim$ be the equivalence relation on $\mathbb{R}_1 \amalg \mathbb{R}_2$ given by setting $x_1 \sim x_2$ if x_i is the copy of x in $\mathbb{R}_i$ for $i = 1, 2$ and $x \neq 0$. Let $X := \mathbb{R}_1 \amalg \mathbb{R}_2 / \sim$ be the associated quotient space. Prove that X is not Hausdorff.

X is called the 'line with two origins'.

Problem 6: Let $X = \mathbb{R}^{n+1} - \{0\}$ be $(n+1)$-dimensional Euclidean space with the origin removed. Define an equivalence relation $\sim$ on X by the rule $v \sim \lambda v$ for all vectors $v \in X$ and scalars $\lambda \neq 0$. The *n-dimensional real projective space* is the quotient $\mathbb{P}^n(\mathbb{R}) := X/\sim$. Prove that $\mathbb{P}^n(\mathbb{R})$ is Hausdorff.

We can think about $\mathbb{P}^n(\mathbb{R})$ as the set of lines through the origin in $\mathbb{R}^{n+1}$. We can define complex projective space $\mathbb{P}^n(\mathbb{C})$ in the same way; it is also Hausdorff.

Problem 7: Let D^n be the closed n-dimensional unit disc

$$D^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, x_1^2 + \cdots + x_n^2 \leq 1\}$$

and let S^{n-1} be its boundary. Exhibit a continuous bijection $f : D^n/S^{n-1} \to S^n$. (We will soon have the technology to prove that any such continuous bijection f must be a homeomorphism – S^n is Hausdorff and D^n/S^{n-1} is compact.)

Problem 8: Let M be the Möbius strip and let $D = D^2$ be the closed 2-dimensional disc. Prove that the quotient space $X = (M \amalg D)/\sim$ obtained by glueing the boundary circle of M to the boundary circle of D is homeomorphic to 2-dimensional real projective space $\mathbb{P}^2(\mathbb{R})$. (Hint: Model $\mathbb{P}^2(\mathbb{R})$ as the sphere S^2 with antipodal points identified. Constructing the actual maps here is tedious; intuitive 'cut and paste arguments' are OK. Consider the 'arctic and antarctic circles' on S^2.)

Problem 9: (Optional - not to be handed in.) Let $k \leq n$ be positive integers. The *(real) Grassmannian* $Gr(k,n)$ is the family of all k-dimensional subspaces of $\mathbb{R}^n$:

$$Gr(k,n) = \{V \text{ a subspace of } \mathbb{R}^n \,:\, \dim(V) = k\}.$$

(a) Explain why $Gr(1,n)$ is the same as projective space $\mathbb{P}^{n-1}(\mathbb{R})$.

We give $Gr(k,n)$ a topology as follows. Let $\mathcal{U}_{k,n} \subset \mathrm{Mat}_{k\times n}(\mathbb{R})$ be the family of full rank $k \times n$ real matrices.

(b) Prove that $\mathcal{U}_{k,n}$ is an open subset of $\mathrm{Mat}_{k\times n}(\mathbb{R})$. (Hint: Having full rank means something about the minors, i.e. sub-determinants, of M.)

The group $GL_k(\mathbb{R})$ acts on $\mathcal{U}_{k,n}$ by the rule $A.M = AM$ for all $A \in GL_k(\mathbb{R})$ and $M \in \mathcal{U}_{k,n}$.

(c) Prove that $Gr(k,n)$ may be identified with the set of *orbits*

$$GL_k(\mathbb{R})\backslash\mathcal{U}_{k,n} := \{GL_k(\mathbb{R}).M \,:\, M \in \mathcal{U}_{k,n}\}.$$

(Hint: Linear algebra.) We may new endow $Gr(k,n)$ with the *quotient topology.*

A *partition* is a way to shove boxes into a corner. We will consider shoving 1×1 boxes into the northeast corner of a $k \times (n-k)$ room (this corner choice is called *Japanese notation*). If we have a matrix $M \in \mathcal{U}_{n,k}$, there is a unique matrix $A \in GL_k(\mathbb{R})$ such that AM is in *row reduced echelon form* (why?). If $n = 9$ and $k = 4$, such a RREF matrix might look like this:

$$\begin{pmatrix} 0 & 1 & * & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}$$

As you can see, the stars in the above picture form a (Japanese) partition:

$$\begin{array}{cccc} * & * & * & * \\ & & * & * \\ & & * & * \\ & & & * \end{array}$$

We might call this partition $(4,2,2,1)$. The *open Schubert cell* $C_\lambda \subset Gr(k,n)$ corresponding to a partition $\lambda \subset k \times (n-k)$ is the family of all orbits which can be represented by matrices with RREF corresponding to λ.

(d) Prove that we have a disjoint union decomposition $Gr(k,n) = \bigsqcup_{\lambda \subset k\times(n-k)} C_\lambda$.

(e) Prove that C_λ is homeomorphic to $\mathbb{R}^{|\lambda|}$, where $|\lambda|$ is the number of boxes in λ. (This requires ingenuity – how can you write down your homeomorphism?)