Math 190: Winter 2018
Homework 6
Due 5:00pm on Thursday 3/1/2018

Problem 1: A topological space X is called *totally disconnected* if the only connected components of X are singletons. Prove that $\mathbb{R}_\ell$ is totally disconnected.

Problem 2: Let X be a compact Hausdorff space. Prove that X is *regular*. This means that, given a closed set $A \subset X$ and a point $x \notin A$, there exist open sets $U, V \subset X$ such that $x \in U, A \subset V$, and $U \cap V = \varnothing$.

Problem 3: Let X be an ordered set in the order topology such that for all $a < b$ in X, the interval $[a, b]$ is compact. Prove that X has the least upper bound property (i.e. every subset $S \subset X$ which is nonempty and bounded above has a least upper bound).

Problem 4: Is $[0, 1]$ compact as a subspace of $\mathbb{R}_\ell$?

Problem 5: Let $f : X \to Y$ be a surjective continuous function between topological spaces. If X is locally path connected, is Y necessarily locally path connected?

Problem 6: Prove that $\mathbb{Q}$ is not locally compact in the standard topology.

Problem 7: Let $f : X \to Y$ be a surjective continuous function between topological spaces. If X is locally compact, is Y necessarily locally compact?

Problem 8: Prove that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to S^n.

Problem 9: (Optional - not to be handed in.) Let R be a commutative ring (with a unit $1 \neq 0$). The *prime spectrum* $\mathrm{Spec}(R)$ is

$$\mathrm{Spec}(R) := \{P \subset R \,:\, P \text{ is a prime ideal}\}.$$

Describe $\mathrm{Spec}(\mathbb{Z})$ and $\mathrm{Spec}(\mathbb{C}[x])$.

Given an ideal $I \subset R$, define $\mathbf{V}(I) \subset \mathrm{Spec}(R)$ by

$$\mathbf{V}(I) := \{P \in \mathrm{Spec}(R) \,:\, I \subset P\}.$$

Prove that the sets $\mathbf{V}(I)$, as I varies over all possible ideals of R, form the closed sets for a topology on $\mathrm{Spec}(R)$. This is called the *Zariski topology*.

Now let $\varphi : R \to S$ be a ring homomorphism. Describe how φ induces a map $\varphi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$. (Notice the direction reversal!) Prove that φ^* is continuous.