Math 190 — Lecture 1 — 1/8/2018

HW : 25% M1 : 20% (2/2) M2 : 20% (3/2) F : 35% (3/19, 11:30–2:29)

OH : MWF 10:00–10:50 ~~APM~~ am , 7250 APM
or by appointment

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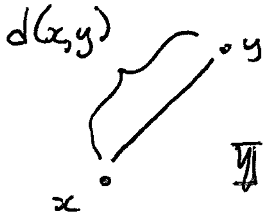
Topology is the study of <u>space</u>.

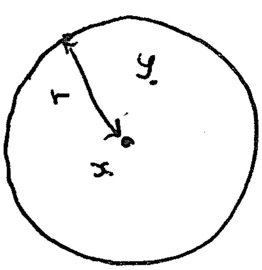
<u>Ex</u> $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$. The Euclidean/<u>standard</u> metric is

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}, \quad d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{1}{2}}.$$



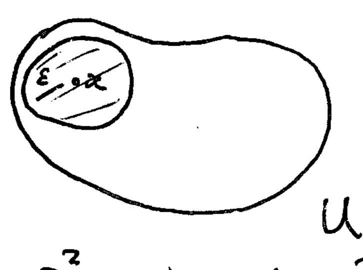
~~A subset $U \subseteq \mathbb{R}^n$ is open if $\forall x \in U$,~~

Given $x \in \mathbb{R}^n$ and $r > 0$, the open <u>ball</u> of radius r centered at x is $B(x,r) = \{y \in \mathbb{R}^n : d(x,y) < r\}$.

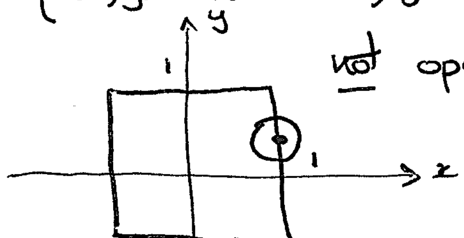


A subset $U \subseteq \mathbb{R}^n$ is <u>open</u> if for all $x \in U$, $\exists\ \varepsilon > 0$ st $B(x, \varepsilon) \subseteq U$.

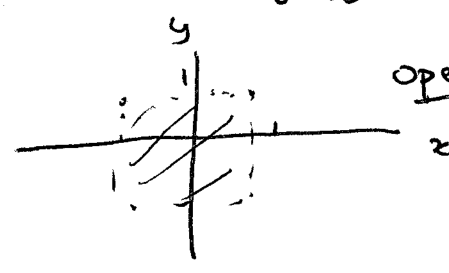
(<u>Rmk</u> $A \subseteq B$ means $x \in A \Rightarrow x \in B$.) $A = B$ ok!



eg. $\{(x,y) \in \mathbb{R}^2 : |x|, |y| \leq 1\}$ <u>not</u> open



but $\{(x,y) \in \mathbb{R}^2 : |x|, |y| < 1\}$ <u>open</u>.



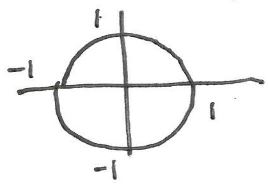
Ex $\mathbb{R}^n$ has interesting subspaces:

- The unit n-dimensional sphere $S^n \subseteq \mathbb{R}^{n+1}$

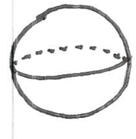
$$S^n = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + \dots + x_n^2 = 1\}.$$

$\mathbb{R}^2$

S^0



S^1



S^2

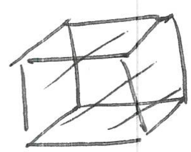
etc.

- The unit interval $I = [0,1] \subseteq \mathbb{R}$

- The n-dim'l hypercube $I^n = \underbrace{I \times \dots \times I}_{n} \subset \mathbb{R}^n$

I^1

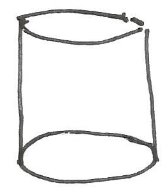
I^2



I^3

etc.

The cylinder $S^1 \times I \subseteq \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$

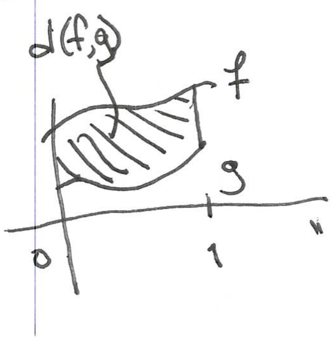


etc.

Ex $\mathcal{C}([0,1]) = \{$ all continuous fcns $f: [0,1] \to \mathbb{R}\}$.

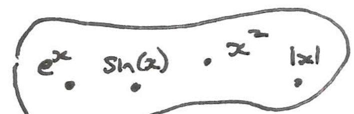
has a metric $d: \mathcal{C}([0,1]) \times \mathcal{C}([0,1]) \longrightarrow \mathbb{R}_{\geq 0}$

$$(f, g) \longmapsto \int_0^1 |f(x) - g(x)|\, dx.$$



This gives $\mathcal{C}([0,1])$ the structure of a "metric space".

$\mathcal{C}([0,1])$



<u>CLAIM</u> $\mathcal{T}$ is a topology on $\mathbb{R}^n$.

<u>Pf</u> We have $\phi \in \mathcal{T}$ (vacuously) and $\mathbb{R}^n \in \mathcal{T}$ (for all $x \in \mathbb{R}^n$, $B(x,1) \subseteq \mathbb{R}^n$).

Let $\{U_\alpha : \alpha \in I\}$ be a coll'n of sets in $\mathcal{T}$.

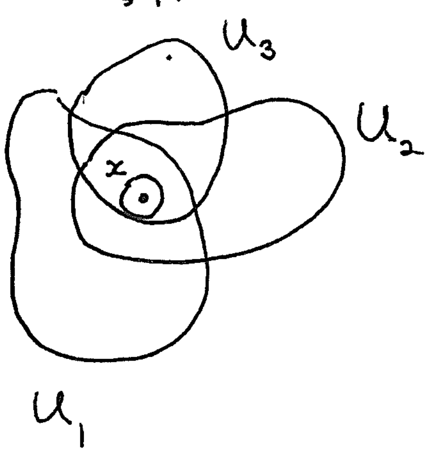
Let $x \in \bigcup_{\alpha \in I} U_\alpha$. $\exists$ $\alpha_0 \in I$ st $x \in U_{\alpha_0}$.

B/c $U_{\alpha_0} \in \mathcal{T}$ $\exists$ $\varepsilon_0 > 0$ st $B(x, \varepsilon_0) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in I} U_\alpha$.

So $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.

Let $\{U_1, -, U_n\}$ be a <u>finite</u> family of sets in $\mathcal{T}$ & let $x \in U_1 \cap \dots \cap U_n$. For $1 \leq i \leq n$ $\exists$ $\varepsilon_i > 0$ s.t. $B(x, \varepsilon_i) \subseteq U_i$. Then if $\varepsilon := \min(\varepsilon_1, -, \varepsilon_n)$ we have $\varepsilon > 0$ and $B(x,\varepsilon) \subset U_1 \cap - \cap U_n$, so $U_1 \cap - \cap U_n \in \mathcal{T}$.

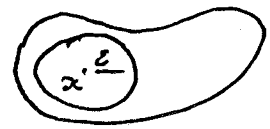
//



Math 190 Lecture 2 1/10/2018

Last Time: Topological Space $(X, \mathcal{T})$ — X: set; $\mathcal{T}$: family of subsets of X

- $\emptyset, X \in \mathcal{T}$
- If $\{U_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$
- If $U_1, \ldots, U_n \in \mathcal{T}$ then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$

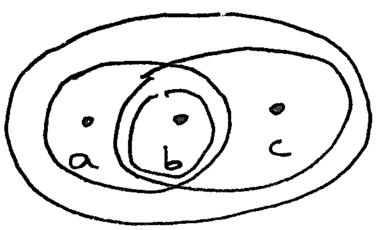
Standard Top. on $\mathbb{R}^n$: U 

$U \in \mathcal{T} \iff \forall\, x \in U\ \exists\ \varepsilon > 0$ st $B(x, \varepsilon) \subset U$.

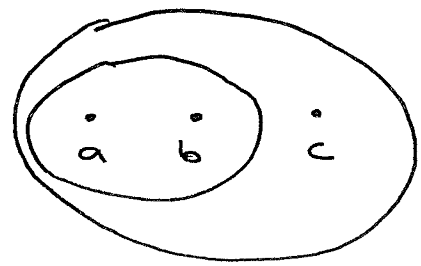
Def Let $\mathcal{T}, \mathcal{T}'$ be topologies on a set X.

We say $\mathcal{T}$ is finer than $\mathcal{T}'$ ($\mathcal{T}'$ is coarser than $\mathcal{T}$) if $\mathcal{T} \supset \mathcal{T}'$.

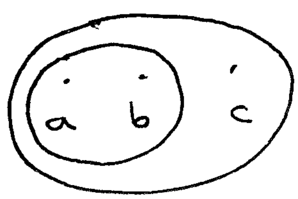
eg $X = \{a, b, c\}$



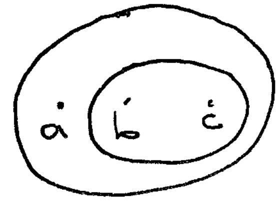
is finer than



but



&



are incomparable.

$X = \mathbb{R}^n$:

(disc. topology) is finer than (Euclidean top) is finer than (indisc. top.)

<u>Ex</u> X - any set

$\mathcal{T} = \{ U \subset X : X - U$ is ~~empty~~ X or finite $\}$

"finite complement topology"

<u>CLAIM</u> $\mathcal{T}$ <u>is</u> a topology on X.

- $X - X = \emptyset$ so $X \in \mathcal{T}$
- $X - \emptyset = X$ so $\emptyset \in \mathcal{T}$.

If $U_\alpha \in \mathcal{T}$ for all $\alpha \in I$, $X - \left(\bigcup_\alpha U_\alpha\right) = \bigcap_\alpha (X - U_\alpha)$

so that $\bigcup_\alpha U_\alpha \in \mathcal{T}$ (any ~~an~~ intersection of finite sets is finite!)

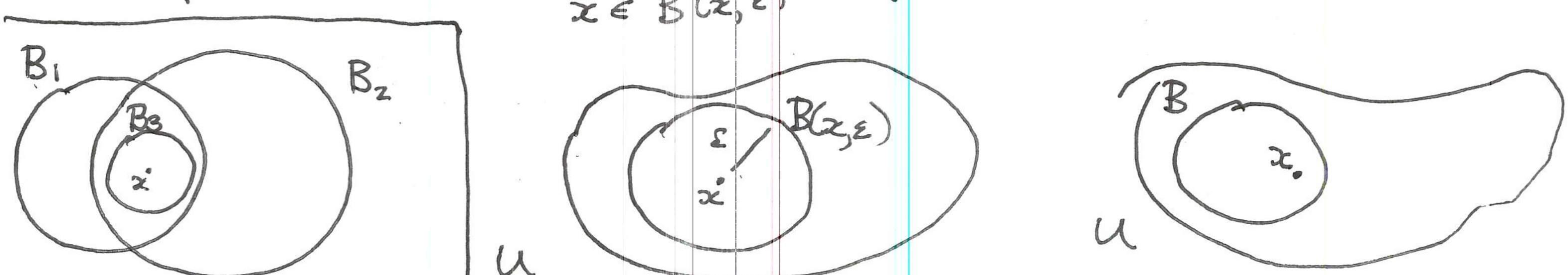
If $U_1, \ldots, U_n \in \mathcal{T}$, $X - \left(\bigcap_{i=1}^n U_i\right) = \bigcup_{i=1}^n (X - U_i)$

so that $\bigcap_{i=1}^n U_i \in \mathcal{T}$ (a <u>finite</u> union of finite sets is finite!). //

§13 <u>Bases</u> <u>for</u> <u>topologies</u>

<u>Idea</u> $\mathbb{R}^n$, std top.

$U \subset \mathbb{R}^n$ is open $\overset{\text{def}}{\iff}$ $\forall x \in U \ \exists \ \varepsilon > 0$ st $x \in B(x, \varepsilon) \subset U$. $\overset{\text{check!}}{\iff}$ $\forall x \in U \ \exists$ some open ball $B \subset \mathbb{R}^n$ st $x \in B \subset U$.



<u>Def</u> Let X be a set, $\mathcal{B}$ a family of subsets of X.

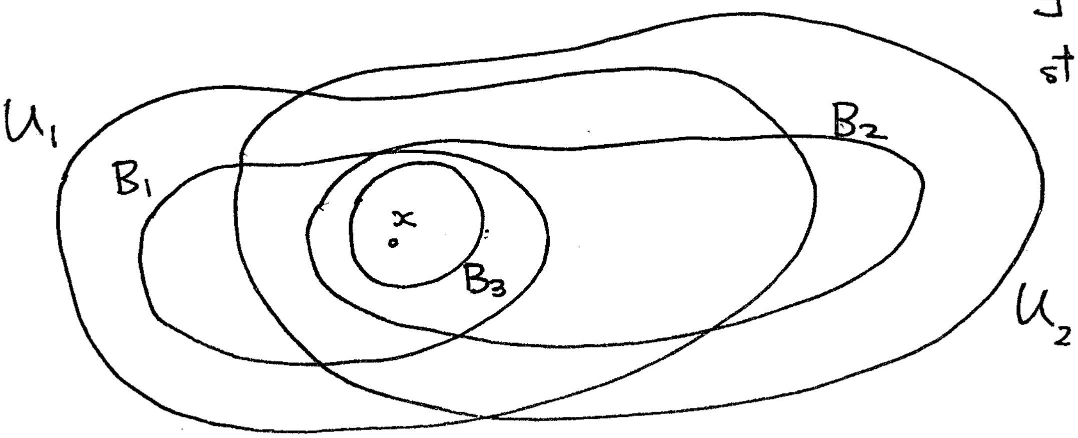
$\mathcal{B}$ is a <u>basis</u> <u>for a</u> <u>topology</u> <u>on</u> <u>X</u> if

1. $\forall x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$.
2. If $B_1, B_2 \in \mathcal{B}$ & $x \in B_1 \cap B_2$, $\exists B_3 \in \mathcal{B}$ st $x \in B_3 \subset B_1 \cap B_2$.

Prop Let $\mathcal{B}$ be a basis for a top. on X & let $\mathcal{T}$ be the top. gen'd by $\mathcal{B}$. Then $\mathcal{T}$ is a top. on X.

Pf

- $\emptyset \in \mathcal{T}$ ✓
- $X \overset{?}{\in} \mathcal{T}$: for all $x \in X$, $\exists$ $B \in \mathcal{B}$ st $x \in B \subset X$, so $X \in \mathcal{T}$.
- Let $U_\alpha \in \mathcal{T}$ for all $\alpha \in I$, let $x \in \bigcup_\alpha U_\alpha$. $\exists$ α_0 s.t. $x \in U_{\alpha_0}$. Then $\exists$ $B \in \mathcal{B}$ st $x \in B \subset U_{\alpha_0} \subset \bigcup_\alpha U_\alpha$. So $\bigcup_\alpha U_\alpha \in \mathcal{T}$.
- Let $U_1, U_2 \in \mathcal{T}$. We show $U_1 \cap U_2 \in \mathcal{T}$. (& apply induction)



$\exists$ $B_1, B_2 \in \mathcal{B}$ st $x \in B_1 \subset U_1$, $x \in B_2 \subset U_2$. So $\exists$ $B_3 \in \mathcal{B}$ st $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$.

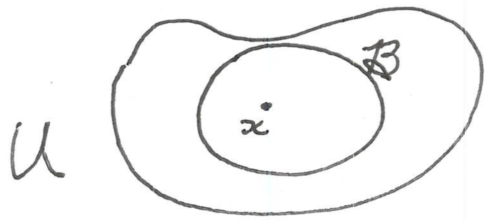
So $U_1 \cap U_2 \in \mathcal{T}$. //

Rmk If $\mathcal{B}$ is a basis for a top. on X, $\mathcal{T}$ is the top gen'd by $\mathcal{B}$, then

$$\mathcal{T} = \{\text{all unions of sets in } \mathcal{B}\}.$$

The topology $\mathcal{T}$ gen'd by $\mathcal{B}$ is given by

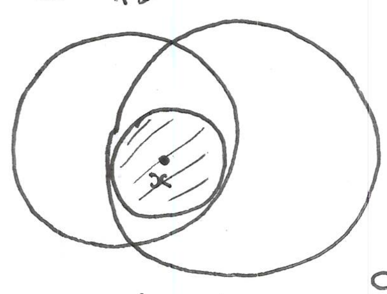
$U \in \mathcal{T} \iff \forall\, x \in U\ \exists\, B \in \mathcal{B} \ni x \in B \subset U.$



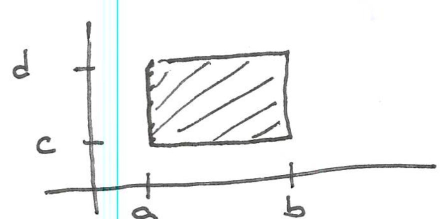
$\{B(x,r) : x \in \mathbb{R}^2,\ r > 0\}.$

<u>Ex</u> $X = \mathbb{R}^2$, $\mathcal{B} = \{\text{all open disks in } \mathbb{R}^2\}$

Then $\mathcal{B}$ is the basis for a topology on ~~X~~ $\mathbb{R}^2$.

- $\forall\, x \in \mathbb{R}^2 \quad x \in B(x,1)$
- 

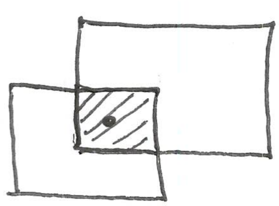
The topology $\mathcal{T}$ gen'd by $\mathcal{B}$ is the Euclidean topology.



~~Fact Let $\mathcal{T}$~~

<u>Ex</u> $X = \mathbb{R}^2$, $\mathcal{B}' = \{(a,b) \times (c,d) : a,b,c,d \in \mathbb{R}\}$

Then $\mathcal{B}'$ is ALSO the basis for a top. on $\mathbb{R}^2$.



<u>CHECK</u> The topology $\mathcal{T}'$ gen'd by $\mathcal{B}'$ is <u>ALSO</u> the Euclidean top on $\mathbb{R}^2$. (Even though $\mathcal{B} \cap \mathcal{B}' = \emptyset$.)