Math 190 Lecture 11 2/12/2018

Last Time • X - set d - metric on X

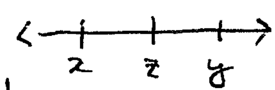
metric topology on X has basis $\{B_d(x,\varepsilon) : x\in X, \varepsilon>0\}$.

A space X is metrizable if $\exists$ a metric d on X s.t.

metric topology on $(X,d)$ = given topology on X. * Metrizable $\Rightarrow$ Hausdorff.

- $\mathbb{R}^\omega = \mathbb{R}\times\mathbb{R}\times\cdots$ not metrizable in box topology.

CLAIM If $(X,<)$ is any ordered set, order top. on X is Hausdorff.




Pf Let $x<y$ in X. Suppose $\exists$ $z\in X$ st $x<z<y$.

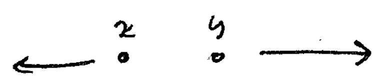
① If $x'<x$ & $y'>y$, $(x',z)\cap(z,y')=\emptyset$

② If $x$ is smallest & $y'>y$, $[x,z)\cap(z,y')=\emptyset$.

③ If $x'<x$ & $y$ is largest, $(x',z)\cap(z,y]=\emptyset$.

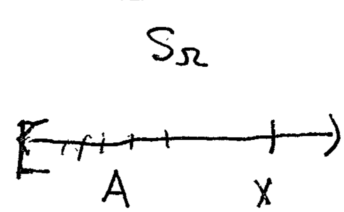
④ If $x$ is smallest, $y$ is largest, $[x,z)\cap(z,y]=\emptyset$.

Suppose $\nexists$ z st $x<z<y$.




① If $x'<x$ & $y'>y$, $(x',y)\cap(x,y')=\emptyset$, ② ③ ④ etc.

Q Is every order topology metrizable? NO.



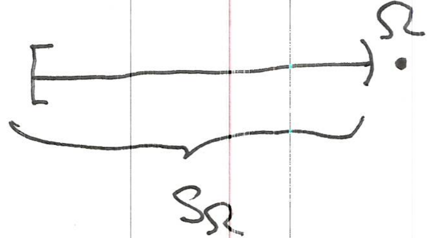

$S_\Omega$ = minimal uncountable well ordered set

• no largest elt,

• if $A\subset S_\Omega$ countable, $\exists$ $x\in S_\Omega$ st $A<x$.

<u>FACT</u> Let $X = S_\Omega \cup \{\Omega\}$ (maximum elt). Then $X$ is <u>NOT</u> metrizable in order top. (but it's Hausdorff!).

<u>Pf</u> $X$




We claim $\Omega \in \overline{S_\Omega}$. Indeed, if $(x, \Omega]$ is any basic nbhd of $\Omega$ then $x \in S_\Omega$ and $\exists\ x' \in S_\Omega$ st $x' > x$. So $x' \in (x, \Omega] \cap S_\Omega$. Thus $\Omega \in \overline{S_\Omega}$.

However, if $(x_n)_{n \geq 1}$ is a sequence in $S_\Omega$ st $x_n \to \Omega$, then $\{x_n : n \geq 1\} \subset S_\Omega$ is <u>countable</u> so $\exists\ z \in S_\Omega$ st $z > x_n$ for all $n \geq 1$. So $(z, \Omega]$ is a nbhd of $\Omega$ & $x_n \notin (z, \Omega]$ for all $n \geq 1$. Thus $x_n \not\to \Omega$. $\divideontimes$

<u>Rmk</u> $X = S_\Omega \cup \{\Omega\}$ is written $\overline{S_\Omega}$.

<u>FACT</u> Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces & let $f: X \to Y$ be a function.

$f$ is CTS $\iff$ for all $x \in X$, for all $\varepsilon > 0$, $\exists\ \delta > 0$ st $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$.

"$\varepsilon / \delta$ - continuity btwn metric spaces!"

<u>FACT</u> Let $X, Y$ be top spaces & $f: X \to Y$ a fcn.

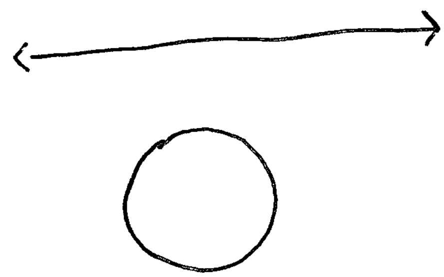
$f$ is CTS $\Rightarrow$ for every sequence $(x_n)$ in $X$ st $x_n \to x$, we have $f(x_n) \to f(x)$ in $Y$.

Converse holds if $X$ is metrizable. "Sequential continuity"

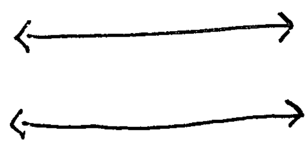
## Disjoint Union Topology

Let $X, Y$ be top. spaces. The disjoint union topology on the disj. union $X \amalg Y$ has open sets $\{U \amalg V : U \subset X \text{ open}, V \subset Y \text{ open}\}$

e.g. $\mathbb{R} \amalg S^1$ :



$\mathbb{R} \amalg \mathbb{R}$



More generally If $\{X_\alpha\}$ is a family of spaces, $\coprod_\alpha X_\alpha$ has open sets $\coprod_\alpha U_\alpha$ for $U_\alpha \subset X_\alpha$ open.

## Universal Property of Disjoint Union

$$\begin{tikzcd}
 & X \arrow[d, hook, "\iota_X \text{ (inclusion)}"] \arrow[ddr, bend left, "f"] & \\
Y \arrow[r, hook, "\iota_Y"'] \arrow[drr, bend right, "g"'] & X \amalg Y \arrow[dr, dashed, "\exists! h"] & \\
 & & Z
\end{tikzcd}$$

arrows are reversed

$$\begin{tikzcd}
 & X & \\
Y & X \times Y \arrow[l, two heads, "\pi_Y"'] \arrow[u, two heads, "\pi_X"] & \\
 & & Z \arrow[ul, dashed, "\exists!"] \arrow[uul, bend right] \arrow[ull, bend left]
\end{tikzcd}$$

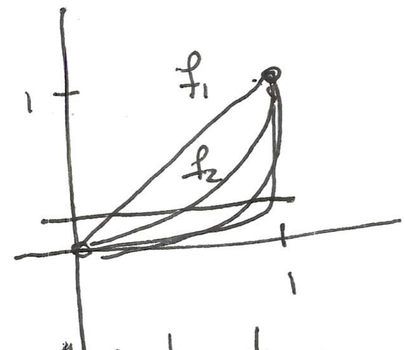
**FACT** Let $X$ be a top. space & let $Y = (Y,d)$ be a metric space. Let $f_n : X \to Y$ $(n=1,2,\dots)$ be a sequence of CTS fcns & let $f : X \to Y$ be a fcn. If

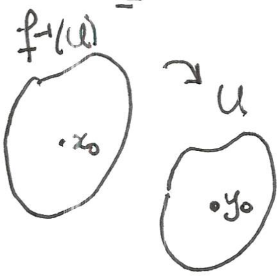
$$\circledast \quad \text{for all } \varepsilon > 0, \ \exists\, N \text{ s.t. } n \geq N \text{ implies } d(f_n(x), f(x)) < \varepsilon \text{ for all } x \in X$$

("$f_n$ converges uniformly to $f$")

then $f$ is CTS.

**Ex** $X = Y = [0,1]$ $\quad f_n(x) = x^n$




$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$ ← "pointwise" limit, not CTS. No uniform convergence!

**Pf** Let $U \subseteq Y$ be open. We want to show that $f^{-1}(U) \subset X$ is open.

Let $x_0 \in f^{-1}(U)$ & let $y_0 = f(x_0) \in U$. $\exists\ \varepsilon > 0$ st $B_d(y_0, \varepsilon) \subset U$.

Choose $N \geq 1$ st $n \geq N \Rightarrow d(f(x), f_n(x)) < \varepsilon/3$ for all $x \in X$.

So $d(f(x), f_N(x)) < \varepsilon/3$ for all $x \in X$. Since $f_N$ is CTS,

$W := f_N^{-1}(B_d(f_N(x_0), \varepsilon/3))$ is a nbhd of $x_0$ in $X$.

We claim $W \subset f^{-1}(B_d(y_0, \varepsilon)) \subset f^{-1}(U)$. For if $x \in W$,

$$d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \ \blacksquare$$

Math 190 Lecture 12 2/9/2018

## The Quotient Topology

$S^1 \times S^1$

**Def** An equivalence relation $\sim$ on a set $X$ is a binary rln

st
1. $x \sim x \quad \forall x \in X$
2. $x \sim y \iff y \sim x \quad \forall x, y \in X$
3. if $x \sim y$ and $y \sim z$ then $x \sim z$. $\forall x, y, z \in X$.

If $x \in X$, the equivalence class is

$$[x] := \{ y \in X : y \sim x \}.$$

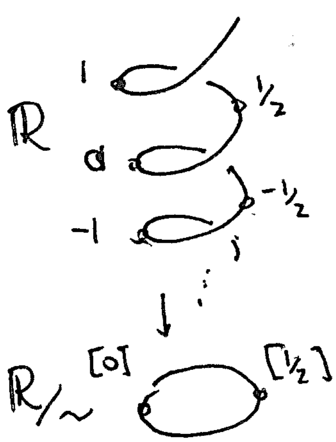
We let $X/\sim \; := \{ [x] : x \in X \}$ be the set of equiv. classes.

**Ex** $X = \mathbb{R}$. Define $x \sim y \iff x - y \in \mathbb{Z}$.

For $x \in \mathbb{R}$, $[x] = \{ \ldots, x-2, x-1, x, x+1, x+2, \ldots \}$

so (eg) $[1] = \mathbb{Z} = [7]$, $[\pi] = [\pi - 4] = \ldots$

We have $\mathbb{R}/\sim \; = \{ [x] : x \in \mathbb{R} \} = \{ [x] : 0 \le x < 1 \}$.




$S^1$??

**Rmk** If $\sim$ is an equiv. rln on $X$, we have a canonical surj'n

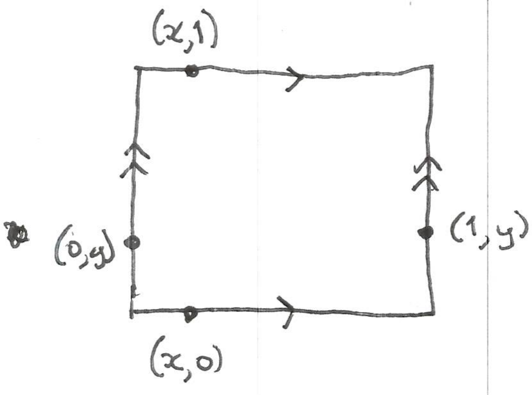
$$\pi : X \longrightarrow X/\sim$$
$$x \longmapsto [x].$$

**Ex** $X = I \times I = \{(x,y) : 0 \leqslant x, y \leqslant 1\}$.

$\sim$ on $X$ given by: $(x,0) \sim (x,1) \quad \forall\ 0 \leqslant x \leqslant 1$

$(1,y) \sim (0,y) \quad \forall\ 0 \leqslant y \leqslant 1$

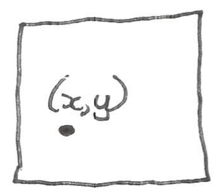
($\sim$ generates an equiv. r'l'n)



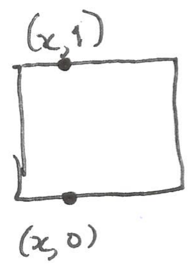


$S^1 \times S^1$ ?

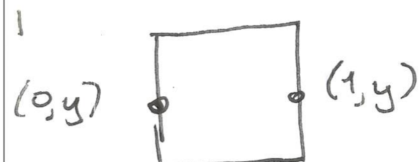
points in $X/\sim$ : $\{(x,y)\}$ for $0 < x, y < 1$



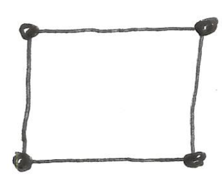

$\{(x,0), (x,1)\}$ for $0 < x < 1$




$\{(0,y), (1,y)\}$ for $0 < y < 1$




$\{(0,0), (0,1), (1,0), (1,1)\}$

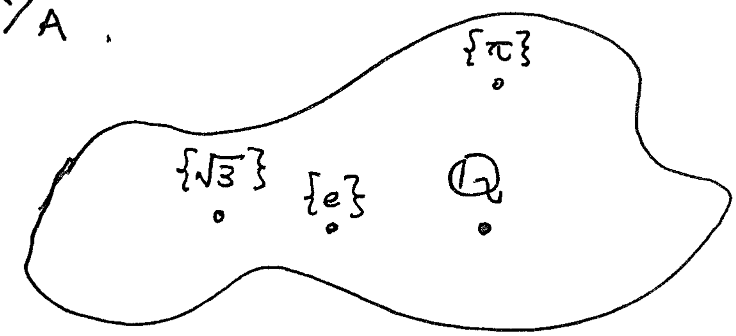


**Ex** $X$ - <u>any</u> set, $A \subset X$ <u>any</u> subset.

$\sim$ on $X$ given by $x \sim y \iff x = y$ or $x, y \in A$.

$$X/_\sim \equiv X/_A .$$

eg $\mathbb{R}/_\mathbb{Q}$

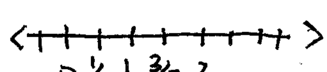
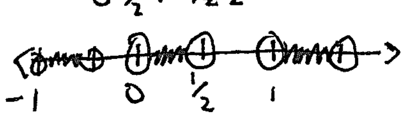



**Q** ~~Hat~~ How to go sets $\longrightarrow$ spaces?

**Def** Let $X$ be a topological space and let $\sim$ be an equivalence rln on $X$. ~~The~~ ~~equ~~ ~~Let $\pi : X \to X/_\sim$, $x \mapsto [x]$ be the canonical surj.~~ The <u>quotient</u> <u>topology</u> on $X/_\sim$ is given by

$$U \subset X/_\sim \text{ is open} \iff \overbrace{\bigcup_{[x] \in U} [x]}^{\pi^{-1}(U)} \subset X \text{ is open.}$$

**Ex** $X = \mathbb{R}$, $x \sim y \iff x - y \in \mathbb{Z}$.

| $U \subset X/_\sim$ | $\pi^{-1}(U) \subset \mathbb{R}$ | $U$ open in $\mathbb{R}/_\sim$? |
|---|---|---|
| $\{[1], [\frac{1}{2}]\}$ |  | No. |
| $\{[x] : 0 < x < \frac{1}{2}\}$ |  | Yes. |

CLAIM The quotient topology on $X/\sim$ is a topology!

Pf

- $\displaystyle\bigcup_{[x]\in X/\sim} [x] = X$, so $X/\sim$ is open.

- $\displaystyle\bigcup_{[x]\in \emptyset} [x] = \emptyset$, so $\emptyset \subset X/\sim$ is open.

- If $\{U_\alpha\}$ is a family of opens in $X/\sim$ then

$$\bigcup_{[x]\in \bigcup_\alpha U_\alpha} [x] = \bigcup_\alpha \underbrace{\left(\bigcup_{[x]\in U_\alpha} [x]\right)}_{\text{open in } X}$$

is open in $X$,

so $\displaystyle\bigcup_\alpha U_\alpha$ is open in $X/\sim$.

- If $U_1, \dots, U_n \subset X/\sim$ are open,

$$\bigcup_{[x]\in \bigcap_{i=1}^n U_i} [x] = \bigcap_{i=1}^n \underbrace{\left(\bigcup_{[x]\in U_i} [x]\right)}_{\text{open in } X}$$

is open in $X$,

so $\displaystyle\bigcap_{i=1}^n U_i$ is open in $X/\sim$. ∎

| $U \subset \mathbb{R}/\mathbb{Q}$ | $\pi^{-1}(U) \subset \mathbb{R}$ | $U \subset \mathbb{R}/\mathbb{Q}$ open? |
|---|---|---|
| $U = \mathbb{R}/\mathbb{Q} - \{\{\sqrt{2}\}\}$ |  | Yes! |
| $U = \mathbb{R}/\mathbb{Q} - \{\mathbb{Q}\}$ | $\mathbb{R}-\mathbb{Q}$  | No! |

<u>Fact</u> If $\sim$ is an equivalence rln on a space $X$, the surj'n $\pi : X \twoheadrightarrow X/\sim$ is continuous.

$$x \longmapsto [x]$$

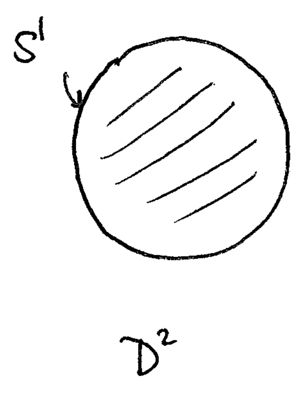
<u>Pf</u> $\mathcal{U} \subset X/\sim$ is open in $X/\sim$ $\Rightarrow$

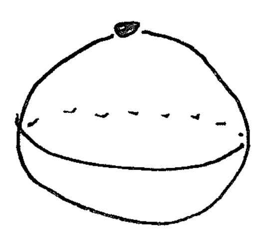
$$\pi^{-1}(\mathcal{U}) = \bigcup_{[x] \in \mathcal{U}} [x] \subset X$$ is open in $X$. //

<u>Q</u> How do we show that $X/\sim \underset{\text{homeom.}}{\cong} Y$?

eg $X = D^2 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$

$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$



$D^2/S^1$ homeo. to $S^2$?

Math 190 Lecture 13 2/12/2018

**Last Time** $X$ = top. space

$\sim$ = equiv rln on $X$ $\quad X/_\sim = \{[x] : x \in X\}$.

**Quotient Topology** on $X/_\sim$

$$\mathcal{U} \subset X/_\sim \text{ open in } X/_\sim \iff \bigcup_{[x]\in \mathcal{U}} [x] \text{ is open in } X.$$

**Fact** This is a topology on $X/_\sim$!

**Pf** • $\displaystyle\bigcup_{[x]\in X/_\sim} [x] = X$, so $X/_\sim$ is open.

• $\displaystyle\bigcup_{[x]\in \emptyset} [x] = \emptyset$, so $\emptyset$ is open.

• If $\{\mathcal{U}_\alpha\}$ is a family of opens in $X/_\sim$,

$$\bigcup_{[x]\in \bigcup_\alpha \mathcal{U}_\alpha} [x] = \bigcup_\alpha \underbrace{\left[\bigcup_{[x]\in \mathcal{U}_\alpha} [x]\right]}_{\text{open in } X \text{ for all } \alpha},$$ ~~we so~~ which is open in $X$.

So $\displaystyle\bigcup_\alpha \mathcal{U}_\alpha$ is open in $X/_\sim$.

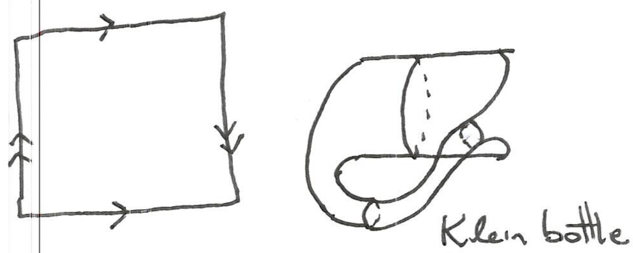
• If $\{\mathcal{U}_1, \ldots, \mathcal{U}_n\}$ is a finite family of opens in $X/_\sim$,

$$\bigcup_{[x]\in \bigcap_i \mathcal{U}_i} [x] = \bigcap_{i=1}^n \underbrace{\left[\bigcup_{[x]\in \mathcal{U}_i} [x]\right]}_{\text{open in } X \text{ for all } i},$$ which is open in $X$.

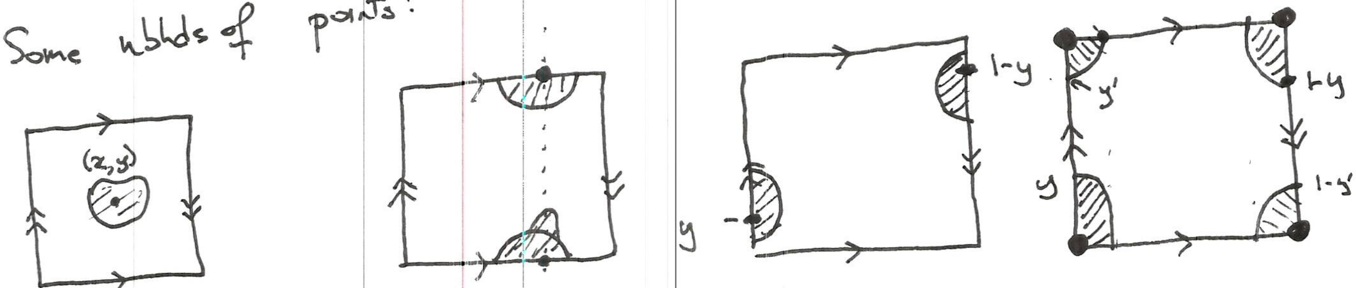
So $\displaystyle\bigcap_{i=1}^n \mathcal{U}_i$ is open in $X$. //

**Ex** $X = I \times I$

$\sim$ : $(x,0) \sim (x,1)$
$(0,y) \sim (0,1-y)$




Some nbhds of points:

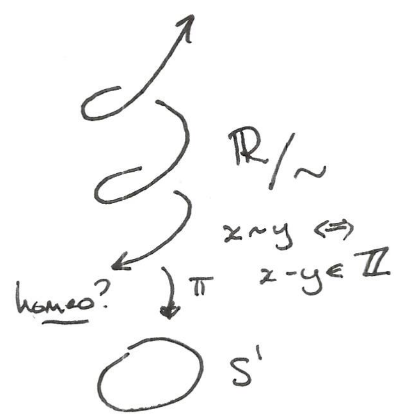



$0 < x, y < 1$ $\qquad$ $0 < x < 1$ $\qquad$ $0 < y < 1$

**Fact** The projn $\pi : X \twoheadrightarrow X/\sim$ is a continuous map.
$x \longmapsto [x]$

⌜ Why? $U \subset X/\sim$ open $\Rightarrow$ $\pi^{-1}(U) = \bigcup_{[x] \in U} [x] \subset X$ is open! ⌟

## Universal Property of Quotient Topology




**Problem** . How to show $X/\sim \underset{\text{homeo.}}{\cong} Y$ ?

. How to get cts maps $f : X/\sim \to Y$?

**Back to sets:**

Let $X, Y$ be sets & $\sim$ an equiv. rln on $X$.
How to get a fcn $f : X/\sim \to Y$.

eg $\sim$ on $\mathbb{R}$ : $x \sim y \iff x - y \in \mathbb{Z}$

$$f : \mathbb{R}/\sim \longrightarrow S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$$
$$[x] \longmapsto e^{\pi i x}$$

BAD!
$[0] = [1]$ in $\mathbb{R}/\sim$ but
$e^{\pi i 0} = 1$ &
$e^{\pi i \cdot 1} = -1$.

<u>Fact</u> Let $X, Y$ be sets and let $\sim$ be an equiv. rln on $X$. Let $\tilde{f} : X \to Y$ be a fcn. The fcn $f : X/_\sim \to Y$, $[x] \mapsto \tilde{f}(x)$ is well-defined iff $x = x' \Rightarrow \tilde{f}(x) = \tilde{f}(x')$. In this case the diagram

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ X/_\sim \arrow[r, dashed, "\exists f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} x \arrow[d, mapsto] & \\ {[x]} \arrow[r, mapsto] & \tilde{f}(x) \end{tikzcd}$$

commutes.

<u>eg</u> $f : \mathbb{R}/_\sim \to S^1$, $[x] \mapsto e^{4\pi i x}$ is well defined b/c

for $x, y \in \mathbb{R}$,

$$x \sim y \iff x - y \in \mathbb{Z} \Rightarrow e^{2\pi(x-y)i} = 1 \Rightarrow e^{4\pi(x-y)i} = 1$$

$$\Rightarrow e^{4\pi i x} = e^{4\pi i y}. \quad //$$

<u>Q</u> What about <u>spaces</u>?

<u>Theorem</u> (Universal Property of $X/_\sim$)

Let $X$ be a space, $\sim$ an equivalence rln on $X$, give $X/_\sim$ the quotient top. & let $\pi : X \twoheadrightarrow X/_\sim$, $x \mapsto [x]$ be the canonical projection. Let $Y$ be another space & let $f : X \to Y$ be a CTS fcn. If $x \sim x' \Rightarrow f(x) = f(x')$, the induced fcn $\bar{f} : X/_\sim \to Y$, $[x] \mapsto f(x)$ is <u>also</u> <u>CTS</u>.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ X/_\sim \arrow[r, dashed, "\bar{f}"'] & Y \end{tikzcd}.$$

Pf Let $U \subset Y$ be open. Then

$$\bar{f}^{-1}(U) \text{ is open in } X/_\sim \iff \bigcup_{[x] \in \bar{f}^{-1}(U)} [x] \text{ is open in } X$$

$$\iff f^{-1}(U) \text{ is open in } X,$$

which is true b/c $f$ is CTS. ▨

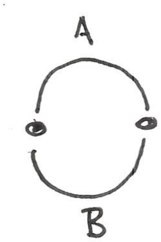
Ex $X = \mathbb{R}$, $x \sim y \iff x - y \in \mathbb{Z}$, $Y = S^1$.

CLAIM $\mathbb{R}/_\sim$ is homeom. to $S^1$

Pf Define $f : \mathbb{R} \longrightarrow S^1$ by $f(x) = e^{2\pi i x}$. $f$ is CTS b/c calculus.

If $x \sim y$ in $\mathbb{R}$ then $f(x) = e^{2\pi i x} = e^{2\pi i x} \cdot \overbrace{e^{2\pi i (y-x)}}^{1} = e^{2\pi i y} = f(y)$

so $f$ induces a fcn $\bar{f} : \mathbb{R}/_\sim \longrightarrow S^1$, $[x] \longmapsto e^{2\pi i x}$. By Universal Property of Quotient Topology, $\bar{f}$ is continuous.




Write $S^1 = \{ z \in \mathbb{C} : |z| = 1 \} = A \cup B$ where

$A = \{ z \in S^1 : \operatorname{Im} z \geq 0 \}$, $B = \{ z \in S^1 : \operatorname{Im} z \leq 0 \}$

Define $g_A : A \longrightarrow \mathbb{R}/_\sim$, $e^{i\theta} \longmapsto [\theta/2\pi]$, $0 \leq \theta \leq \pi$ & $g_B : B \longrightarrow \mathbb{R}/_\sim$, $e^{i\theta} \longmapsto [\theta/2\pi]$, $\pi \leq \theta \leq 2\pi$.

Then $g_A, g_B$ are CTS "b/c calculus" (& $\pi : \mathbb{R} \twoheadrightarrow \mathbb{R}/_\sim$ is CTS).

Also, $g_A|_{A \cap B} = g_B|_{A \cap B}$ (since $[0] = [1]$ in $\mathbb{R}/_\sim$). B/c $A, B \subset S^1$ are closed, Pasting Lemma implies that $g : S^1 \longrightarrow \mathbb{R}/_\sim$, $e^{i\theta} \longmapsto [\theta/2\pi]$, $0 \leq \theta \leq 2\pi$ is CTS.

If $e^{i\theta} \in S^1$ $(0 \leq \theta < 2\pi)$, $\bar{f} g(e^{i\theta}) = \bar{f}\left[\frac{\theta}{2\pi}\right] = e^{2\pi i \frac{\theta}{2\pi}} = e^{i\theta}$. If $[x] \in \mathbb{R}/_\sim$, $g(\bar{f}(x)) = g(e^{2\pi i x}) = [2\pi x / 2\pi] = [x]$.

So, $f$ and $g$ are mutually inverse homeomorphisms and $\mathbb{R}/\sim$ is homeo. to $S^1$. $\blacksquare$

**Math 190** **Lecture 14** **2/14/2018**

**Last Time** $X, Y$ - spaces $\sim$ = equiv rln on $X$

$\underset{\text{CTS}}{f : X \to Y}$ s.t. $x \sim x' \Rightarrow f(x) = f(x')$.

**Universal Property of Quotient Topology** The map $\bar{f} : X/\sim \to Y$, $[x] \mapsto f(x)$

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ X/\sim \arrow[r, dashed, "\bar{f}"'] & Y \end{tikzcd}$$

is CTS.

**Quotient Maps**

**Def** Let $X, Y$ be topological spaces. A surjective fcn $f : X \to Y$ is a *quotient map* if $U \subset Y$ is open $\iff f^{-1}(U) \subset X$ is open.

(So quotient map $\Longrightarrow$ continuous map.)

**Ex** $\pi_Y : X \times Y \to Y$, $(x, y) \mapsto y$ is a quotient map.

*Why?* $\pi_Y^{-1}(U) = X \times U$. So need to show

$X \times U$ is open in $X \times Y \iff U$ is open in $Y$.

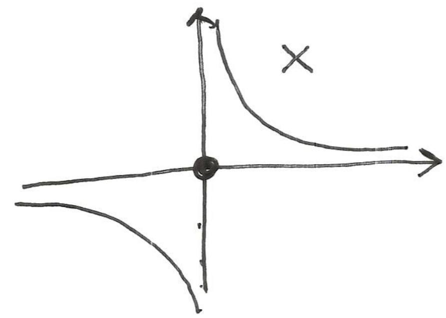
$\Leftarrow$ ✓

$\Rightarrow$ $\forall\ (x,u) \in X \times U\ \exists\ x \in V_{x,u} \underset{\text{open}}{\subset} X$, $u \in W_{x,u} \underset{\text{open}}{\subset} Y$ st $(x,u) \in V_x \times W_u \subset X \times U$. Now (check!) $U = \bigcup_{(x,u) \in X \times U} W_{x,u}$. //

However...

<u>Ex</u> $X = \{xy = 1\} \cup \{(0,0)\} \subset \mathbb{R}^2$

$Y = \mathbb{R}$




$f: X \longrightarrow Y$ is a CTS surjn <u>but</u>
$(x,y) \longmapsto x$

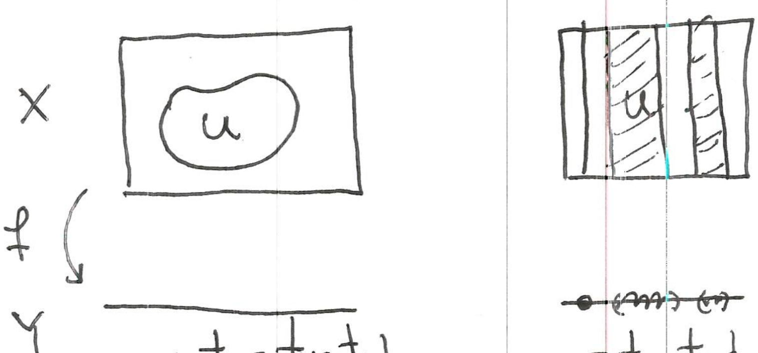
$f^{-1}(\{0\}) = \{(0,0)\}$

$\{0\}$: not open in $Y = \mathbb{R}$ $\qquad$ $(0,0)$: open in $X$.

So $f$ is <u>NOT</u> a quotient map!

<u>Rmk</u> If $\sim$ is an equiv. rln on $X$, $\pi: X \twoheadrightarrow X/\sim$ is a quotient map.

$*$ If $f: X \to Y$ is a function, a subset $U \subset X$ is <u>saturated</u> if for all $x \in U$, $f^{-1}(f(x)) \subset U$.




<u>not saturated</u> $\qquad$ <u>saturated</u>

<u>Rmk</u> Let $f: X \to Y$ be a surjective CTS map.

$f$ is a quotient map $\iff$ for all <u>saturated</u> <u>open</u> sets $U \subset X$, $f(U) \subset Y$ is open.

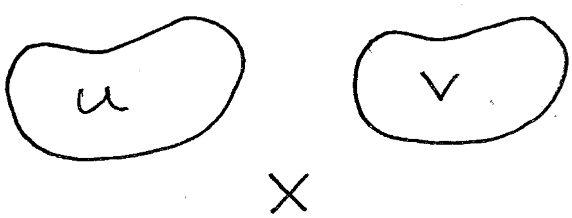
~~Math 199~~ ~~Lecture~~

## §23 Connectedness

**Def** Let $X$ be a space. A separation of $X$ is a pair $U, V \subset X$ of open sets s.t. $U, V \neq \emptyset$, $X = U \cup V$, and $U \cap V = \emptyset$.

$X$ is disconnected if $X$ has a separation.

$X$ is connected otherwise.

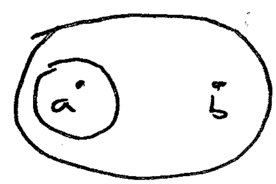



**Ex** · $\mathbb{Q}$ is not connected:

$U = \{r \in \mathbb{Q} : r < \sqrt{2}\}$

$V = \{r \in \mathbb{Q} : r > \sqrt{2}\}$.

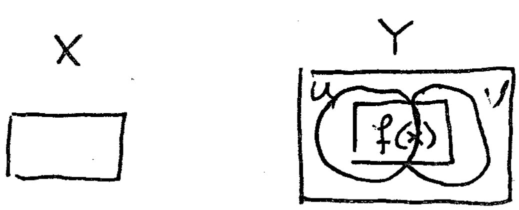
· ~~2~~ $\{a,b\}$ is connected w/ topology




**Q** Is $\mathbb{R}$ connected? $\longleftrightarrow$ Yes...

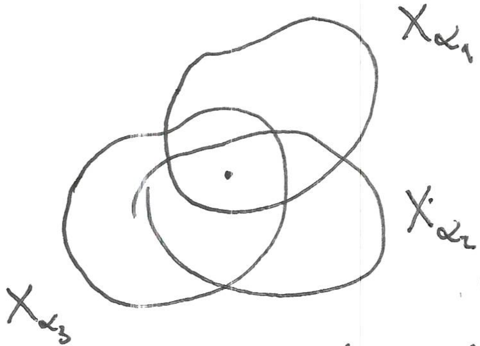
**Fact** Let $f : X \to Y$ be a CTS fcn. If $X$ is connected, then $f(X)$ is connected?

⌈Why? If $f(X) \subseteq U \cup V$ then $X = f^{-1}(U) \cup f^{-1}(V)$. ⌋




**Rmk** If $X$ is homeo. to $Y$ then $X$ is connected $\Longleftrightarrow$ $Y$ is connected.

**Ex** Let $X$ be a space & suppose $X = \bigcup_{\alpha \in I} X_\alpha$ for some connected subspaces $X_\alpha$. If $\bigcap_{\alpha \in I} X_\alpha \neq \emptyset$ then $X$ is connected.




**Pf** Let $x_0 \in \bigcap_{\alpha \in I} X_\alpha$. Suppose $X = U \cup V$ with $U, V \subset X$ open & $U \cap V = \emptyset$. Then $x_0 \in U$ or $x_0 \in V$; WLOG $x_0 \in U$. For all $\alpha \in I$, since $(U \cap X_\alpha) \cup (V \cap X_\alpha) = X_\alpha$ and $x_0 \in U \cap X_\alpha$, This forces $U \cap X_\alpha = X_\alpha$ (b/c $X_\alpha$ is connected). $X_\alpha \subset U$ for all $\alpha$, so $\bigcup_\alpha X_\alpha = X \subset U$ & $X = U$. Thus $V = \emptyset$ & $X$ is connected. ▨

* A subset $U \subset X$ is clopen if it is closed & open.

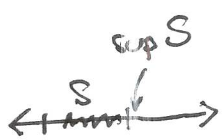
$X$ is connected $\iff$ the only clopen sets $U \subset X$ are $U = \emptyset, X$.

---

**Goal** $\mathbb{R}$, $[0,1]$ etc. are connected.

**Def** An ordered set $L$ is a *linear continuum* if

① for all $x < y$, $\exists\ z \in L$ st $x < z < y$,




② if $S \subseteq L$ is such that $\exists\ x \in L$ st $s < x$ $\forall\ s \in S$, ~~there is~~ $S$ has a least upper bd $\sup S$.

Math 190 Lecture 15 2/16/2018

**Last Time** • $p: X \to Y$ is a quotient map if
- $p$ is surjective, &
- $U \subset Y$ is open $\iff p^{-1}(U) \subset X$ is open.

• $X$ is connected if the only clopen sets $U \subset X$ are $U = X, \emptyset$.

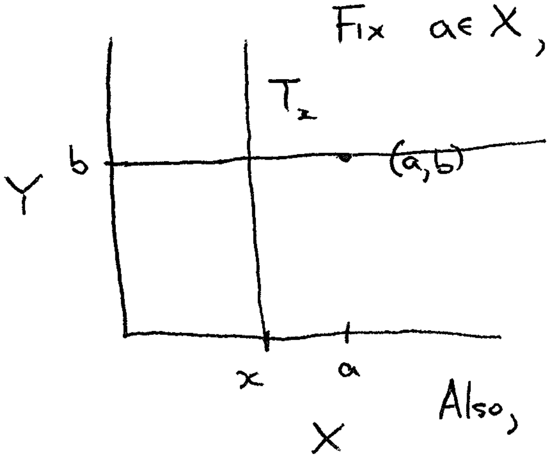
**Rmk** If $X$ is homeom. to $Y$, then $X$ is connected $\iff$ $Y$ is connected.

**Theorem** If $X$ and $Y$ are connected, so is $X \times Y$.

**Pf** For any $x \in X$ or $y \in Y$,

$\{x\} \times Y \underset{\text{homeo.}}{\cong} Y$, $X \times \{y\} \underset{\text{homeo.}}{\cong} X$ are connected.

Fix $a \in X$, $b \in Y$ so $(a,b) \in X \times Y$.




Let $T_x := (\{x\} \times Y) \cup (X \times \{b\})$ for any $x \in X$. Then $(\{x\} \times Y) \cap (X \times \{b\}) \neq \emptyset$ so $T_x$ is connected for each $x \in X$.

Also, $X \times Y = \bigcup_x T_x$ and $\bigcap_x T_x \neq \emptyset$ contains $(a,b)$

so $X \times Y$ is connected. //

⌜$X_1, \dots, X_n$ connected $\Rightarrow X_1 \times \dots \times X_n$ connected.⌟

## §24. Connected subspaces of $\mathbb{R}$.

**Def** An ordered set $X$ has the least upper bd property if for any $\emptyset \neq S \subset X$ st $\exists\, x \in X$ st $\underbrace{s \leq x \ \forall\, s \in S}_{\text{"}S \leq x\text{"}}$, $\exists$ a least upper bd $\sup S$ for $S$.

**Ex** $\mathbb{Q} \leftarrow$ does not have lub.

$\mathbb{Z} \leftarrow$ has lub

$\mathbb{R} \leftarrow$ has lub (axiom...)

**Def** A ~~partial~~ ordered L is a <u>linear</u> <u>continuum</u> if

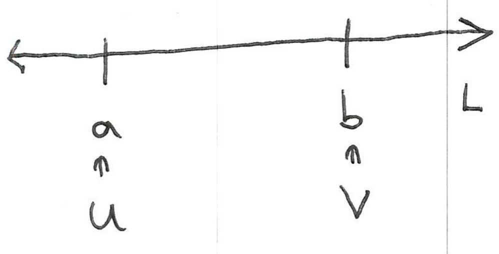
① L has l.u.b. property &

② If $x < y$ in $L$, $\exists$ $z \in L$ st $x < z < y$.

**Theorem** Let $L$ be a linear continuum. Then <u>$L$ is connected</u>, as are subsets of form $(a,b)$ $[a,b)$ $(a,b]$, $[a,b]$, $(a,\infty)$, $[a,\infty)$ $(-\infty,b)$, $(-\infty,b]$. (all l.c.'s!)

**Pf** ~~Let~~ Suppose $U \cup V = L$ is a separation of $L$.

Let $a \in U$ and $b \in V$; WLOG $a < b$.




Let $S = \left\{ x : \begin{matrix} a \leq x \leq b \\ x \in U \end{matrix} \right\}$. Then ~~Let~~

$a \in S$

$S \leq b$, ~~a∈S, sup S~~ so can take

$y := \sup S$. We have $a \leq y \leq b$.

① $y = a$.
Choose $z > a$ s.t. $[a, z) \subset U \subset [a,b]$. $\exists$ $z' \in (a,z)$,
so $z' \in S$ & $y \neq \sup S$. ✱

② $y = b$.
Choose $z < b$ st $(z, b] \subset V \subset [a,b]$. $\exists$ $z' \in (z,b)$,
so $S \leq z'$ & $y \neq \sup S$. ✱

③ $a < y < b$.
If $y \in U$ $\exists$ $(z, z')$ st $y \in (z,z') \subset U$. But $y < z'$ so $y \neq \sup S$. ✱
If $y \in V$ ...

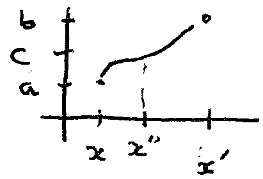
//

**Ex** $\mathbb{R}$, $\mathbb{R}^n$, $[0,1]$, $[0,1]^n$, $S^1$, $S^1 \times \cdots \times S^1$, $S^n$

all connected.

Intermediate Value Theorem

Let $X$ be connected, give $Y$ the order topology & let $f: X \to Y$ be cts. Suppose $x, x' \in X$ are st $a := f(x) < f(x') =: b$. If $c \in Y$ is st $a < c < b$ then $\exists$ $x'' \in X$ st $f(x'') = c$.




**Pf** Otherwise, $X = f^{-1}(-\infty, c) \cup f^{-1}(c, \infty)$ would be a separation of $X$. //

**Ex** Two antipodal pts on Earth have the same temperature

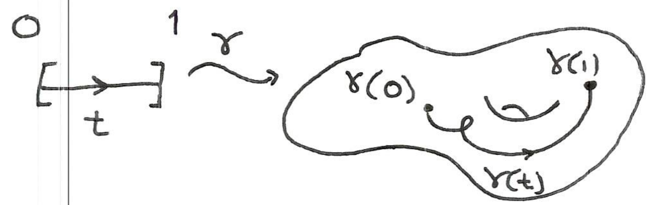
**Pf** Let $T: S^2 \to \mathbb{R}$ be the (cts!) temp. fcn.

Define $f: S^2 \to \mathbb{R}$ by $f(x) = T(x) - T(-x)$.


If $f \equiv 0$, done. Otherwise, $\exists$ $x \in S^2$ st $f(x) \neq 0$, so also $f(-x) = -f(x) \neq 0$. Since $0$ lies btwn $f(x)$, $f(-x) = -f(x)$, & $S^2$ is connected, ~~$\exists x_0 \in S^2$ st~~ by IVT $\exists$ $x_0 \in S^2$ st $f(x_0) = 0$. Thus $T(x_0) = T(-x_0)$. //

Def A **path** in $X$ is a cts fcn

$$\gamma : [0,1] \longrightarrow X.$$




$X$ is **path connected** if for all $x, y \in X$ $\exists$ a path

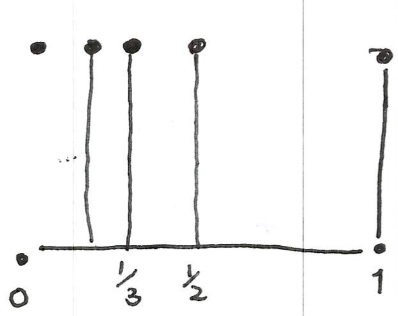
$\gamma : [0,1] \longrightarrow X$ st $\gamma(0) = x$ and $\gamma(1) = y$.

FACT If $X$ is path connected, then $X$ is connected.

Pf Let $X = U \cup V$ be a separation of $X$. If $x \in U$ & $y \in V$, suppose $\gamma : [0,1] \longrightarrow X$ were a path from $x$ to $y$ in $X$.

Then $[0,1] = \gamma^{-1}(U) \cup \gamma^{-1}(V)$ would be a separation of $[0,1]$. But $[0,1]$ is connected. $\divideontimes$

Ex Let $X$ = "topologist's comb" $\subseteq \mathbb{R}^2$ :

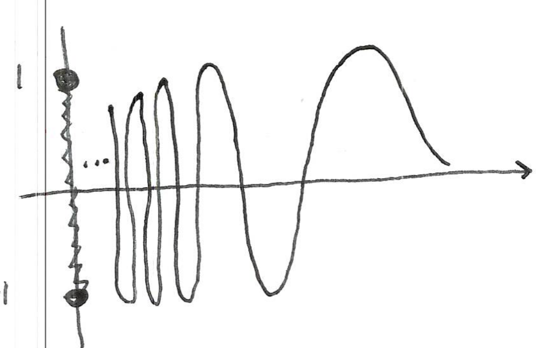



$$X = \{[0,1] \times 0\} \cup \left[\tfrac{1}{n} \times [0,1]\right]_{n=1,2,\ldots} \cup \{(0,1)\}.$$

$X$ is connected but not path connected.

Also topologists sine curve:

$$\left\{ \left(x, \sin\tfrac{1}{x}\right) \right\}_{0<x<1} \cup \{(0,y) : -1 \le y \le 1\}$$



§25

**Def** Let $X$ be a space.

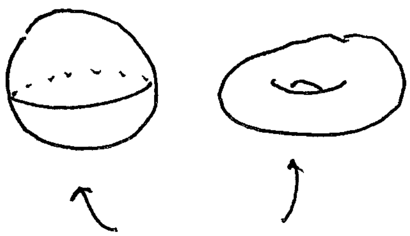
① Define $\sim$ on $X$ by $x \sim y \iff \exists$ a connected subsp. $C \subset X$ st $x, y \in C$.

The equivalence classes of $\sim$ are the (connected) components of $X$.

② Define $\sim$ on $X$ by $x \sim y \iff \exists$ a path $x \longrightarrow y$ in $X$.

The equivalence classes of $\sim$ are the path components of $X$.

$S^2 \amalg (S^1 \times S^1)$

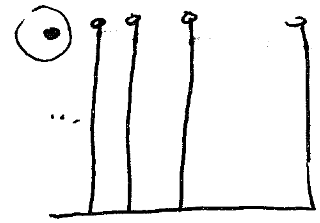


two components

two path components

pc!

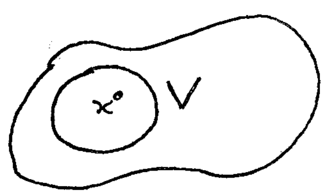
top. comb.



Two pc

One cc.

**Def** A space $X$ is locally connected if for all $x \in X$, for all nbhds $U$ of $x$, $\exists$ a connected nbhd $V$ of $x$ st $x \in V \subset U$.




**Ex**

| $X$ | connected | locally connected |
|---|---|---|
| $\mathbb{R}, S^1, S^n$ | ✓ | ✓ |
| $[0,1] \cup [2,3]$ | × | ✓ |
| $\mathbb{Q}$ | × | × |
| top. comb. | ✓ | × |