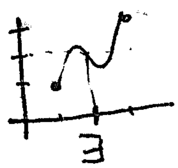
Math 190 Lecture 15 2/21/2018

Last Time • $[0,1]$, $\mathbb{R}$ are connected

(as are $\mathbb{R}^n$, S^n, etc.)

• IVT



~~Def A path~~ **Ex** $[0,1)$ and S^1 are not homeomorphic.

Why? If $f : [0,1) \to S^1$ were a homeom. then f would induce a homeom. $[0,1) - \{\tfrac{1}{2}\} \xrightarrow{\sim} S^1 - \{y\}$

$x \longmapsto f(x)$

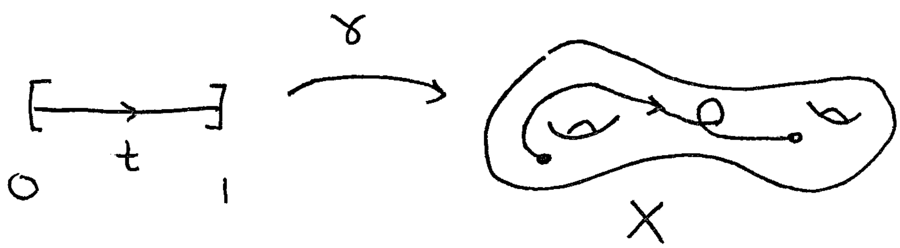
where $y = f(\tfrac{1}{2})$. But $[0,1) - \{\tfrac{1}{2}\}$ is not connected

●—o—o

and $S^1 - \{y\}$

is connected. ┘

Def A path in a space X is a cts fn $\gamma : [0,1] \to X$.



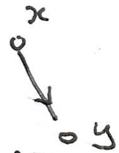
X is path connected if for all $x, y \in X$, $\exists$ a path $\gamma : [0,1] \to X$ st $\gamma(0) = x$, $\gamma(1) = y$.

Eg $\cdot\ \mathbb{R}^n$ $\qquad\cdot\ \mathbb{R}^2 - \{0\}$ $\qquad\cdot\ S^n$

p.c.

"straight line"

$\gamma(t) = ty + (1-t)x$.



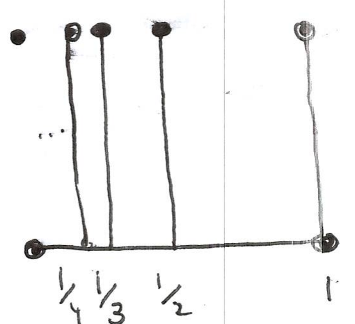
FACT X path-connected $\Longrightarrow$ X connected.

⌜ **Why?** If $X = U \cup V$ is a sep. of X, let $x \in U$, $y \in V$.

If $\gamma : [0,1] \longrightarrow X$ were a path $x \rightsquigarrow y$,

then $[0,1] = \gamma^{-1}(U) \cup \gamma^{-1}(V)$ would be a separation of $[0,1]$, which is connected! ⌟

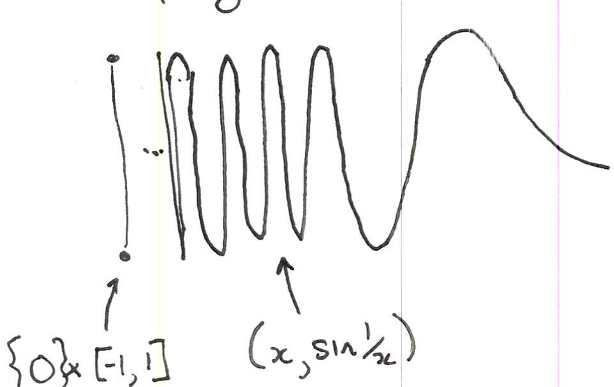
Ex Let X = "topologist's comb"



$$X = \left([0,1] \times \{0\}\right) \cup \left(\{1/n\} \times [0,1]\right)_{n=1,2,3,\dots} \cup \{(0,1)\}.$$

X is connected but NOT path connected.

Also "topologist's sine curve"



Def ~~A~~ Let X be a space.

① Define $\sim$ on X by $x \sim y \iff \exists$ a <u>connected</u> subsp. $Y \subseteq X$ st $x, y \in Y$

② Define $\sim'$ on X by $x \sim y \iff \exists$ a path $x \longrightarrow y$ in X.

Then $\sim$ and $\sim'$ are equiv. rlns on X.

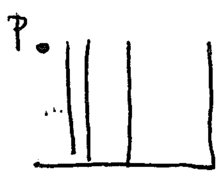
The <u>(connected)</u> <u>components</u> of X are the $\sim$-equiv. classes.

The <u>path</u> <u>components</u> of X are the $\sim'$-equiv. classes.

eg $X = S^1 \sqcup S^2$ components: S^1, S^2

path components: S^1, S^2

$X =$ top. comb.

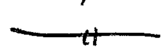


X

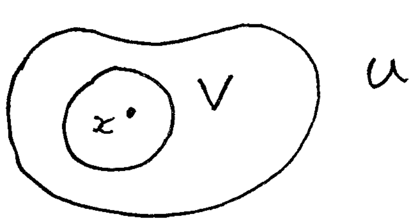
Comps: X

p.c.'s: $X - \{p\}$, $\{p\}$.

$X = \mathbb{Q}$ comps: $\{x\}$, $x \in \mathbb{Q}$

p.c.'s:  .

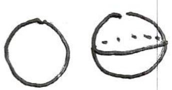
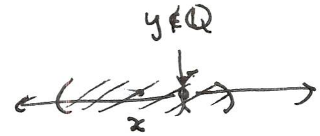
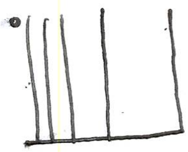
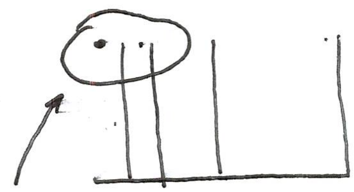
Def Let X be a space. X is <u>locally</u> <u>connected</u> if for every $x \in X$, for all nbhds U of x, $\exists$ a <u>connected</u> nbhd V of x st $x \in V \subset U$.

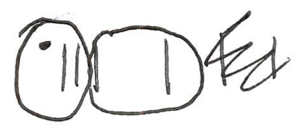


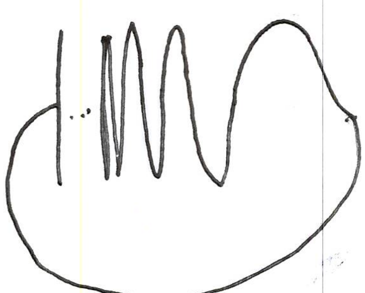
X is <u>locally</u> <u>path</u> <u>connected</u> if...

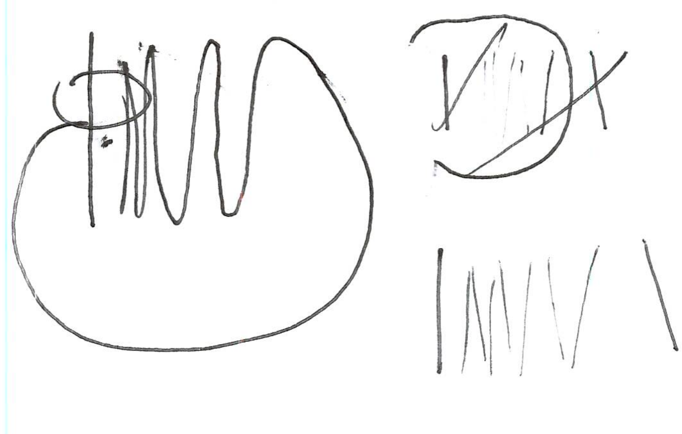
.. <u>path</u> <u>connected</u> nbhd V ...

Ex

X	connected?	locally connected?
$S^1 \sqcup S^2$ 	no.	yes. 
$\mathbb{R}$ 	yes.	yes.
$\mathbb{Q}$	no.	no. 
 top comb	yes.	no. 



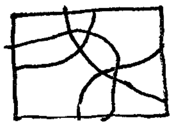
X	path conn.?	locally path connected?
$S^1 \sqcup S^2$	no	yes
$\mathbb{R}$	yes	yes
$\mathbb{Q}$	no	no
$X =$ 	yes	no



§26 Compact Spaces

Def A family of subsets $\mathcal{A}$ of X covers X if $\bigcup_{A \in \mathcal{A}} A = X$.

X



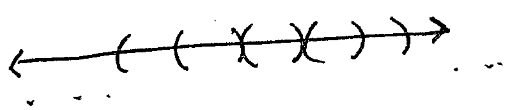
A subcoll'n $\mathcal{A}' \subset \mathcal{A}$ is a subcover if $\bigcup_{A \in \mathcal{A}'} A = X$.

A cover $\mathcal{U}$ of X is open if every set $U \in \mathcal{U}$ is open.

$*$ A topological space X is compact if every open cover $\mathcal{U}$ of X has a finite subcover.

Ex ① $[0,1) = \bigcup_{n \geq 1} [0, 1 - 1/n)$. $\mathcal{U} = \{[0, 1-1/n) : n = 1, 2, \ldots\}$ is an open cover without a finite subcover. $\Rightarrow$ $[0,1)$ not cpct.

② $X = \mathbb{R}$ $\quad \mathcal{U} = \{(n-1, n+1) : n \in \mathbb{Z}\}$ is an open cover w/o a finite subcover $\Rightarrow \mathbb{R}$ not cpct.



③ If $|X| < \infty$ then X is cpct.

④ $[0,1]$, $[0,1]^n$, S^n, $S^1 \times S^1 \times \ldots \times S^1$, ...?

  (CPCT!)

Rmk If X is homeo. to Y, then X cpct $\Leftrightarrow$ Y cpct.

Rmk If $Y \subset X$, a family $\mathcal{A}$ of sets in X covers Y if $Y \subset \bigcup_{A \in \mathcal{A}} A$.

Fact Let ~~If~~ $f: X \to Y$ be a CTS surj'n. If X is cpct, then Y is cpct.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover of Y. Then $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of X (b/c f is CTS)

B/c X is cpct, $\exists\ \alpha_1, \dots, \alpha_n \in I$ st $X = f^{-1}(U_{\alpha_1}) \cup \dots \cup f^{-1}(U_{\alpha_n})$. Since f is surjective, this means $Y = U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$, so $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a finite subcover of $\mathcal{U}$.

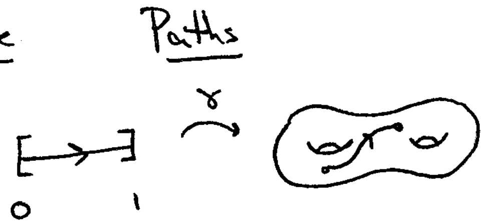
Fact Let $C \subset X$ be a closed subspace. If X is cpct, so is C.

Let $\mathcal{U}$ be an open cover of C, so $\bigcup_{U \in \mathcal{U}} U \supset C$ and U is open in X for all $U \in \mathcal{U}$. Now $\mathcal{U} \cup \{X - C\}$ is an open cover of X (b/c C is closed).

B/c X is cpct, $\exists\ U_1, \dots, U_n \in \mathcal{U}$ st $X = (U_1 \cup \dots \cup U_n) \cup (X - C)$. This means $C \subset (U_1 \cup \dots \cup U_n)$, so $\{U_1, \dots, U_n\}$ is a finite subcover of $\mathcal{U}$.

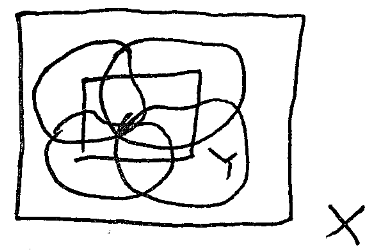
Math 190 | Lecture 17 | 2/23/2018

Last Time **Paths** path connectedness



- X is compact if every open cover $\mathcal{U}$ has a finite subcover.

Rmk If $Y \subset X$ & $\mathcal{U}$ is a family of open sets in X, $\mathcal{U}$ covers Y if $Y \subset \bigcup_{U \in \mathcal{U}} U$.



FACT If X is compact and $Y \subset X$ is closed, then Y is compact.

Pf Let $\mathcal{U}$ be an open cover of Y (w/ U open in X $\forall U \in \mathcal{U}$). Then $\mathcal{U} \cup \{X - Y\}$ is an open cover of X. B/c X is cpct, $\exists$ $U_1, \dots, U_n \in \mathcal{U}$ s.t. $X = (U_1 \cup \dots \cup U_n) \cup (X - Y)$. So $Y \subset U_1 \cup \dots \cup U_n$ & $\{U_1, \dots, U_n\}$ is a finite subcover of Y. ∎

FACT Let X be a Hausdorff space. If $Y \subset X$ is cpct, then Y is closed.

Pf Let $x \in X - Y$; we show $x \notin \overline{Y}$. For all $y \in Y$, $\exists$ nbhds U_y of y, V_y of x st $U_y \cap V_y = \emptyset$. $\{U_y : y \in Y\}$ is an open cover of Y; let $\{U_{y_1}, \dots, U_{y_n}\}$ be a finite subcover. Now $(U_{y_1} \cup \dots \cup U_{y_n}) \cap (V_{y_1} \cap \dots \cap V_{y_n}) = \emptyset$, so $V := V_{y_1} \cap \dots \cap V_{y_n}$ is a nbhd of x st $V \cap Y = \emptyset$. Thus $x \notin \overline{Y}$. ∎

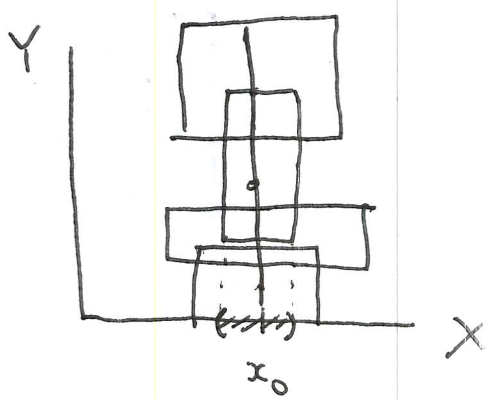
Rmk Let $X = \{a, b\}$ w/ <u>indiscrete</u> topology.

Now $Y = \{a\}$ is cpct but not closed.

FACT If $Y_1, \dots, Y_n$ are cpct subspaces of X, so is $Y_1 \cup \dots \cup Y_n$. (Check!)

FACT If X and Y are cpct, so is $X \times Y$.

Pf



Let $\mathcal{U}$ be an open cover of $X \times Y$; WLOG every set in $\mathcal{U}$ is <u>basic</u>. Consider $x_0 \in X$; then $\{x_0\} \times Y \cong Y$ $(u \times v)$ is cpct. For any $y \in Y$, $\exists$ $U_{x_0,y} \in X$ a nbhd of x_0 & $V_{x_0,y} \subset Y$ a nbhd of y st $U_{x_0,y} \times V_{x_0,y} \in \mathcal{U}$. So

$$\{x_0\} \times Y \subset \bigcup_{y \in Y} (U_{x_0,y} \times V_{x_0,y})$$ & $\exists$ $y_1, \dots, y_n \in Y$ s.t.

$$\{x_0\} \times Y \subset \bigcup_{i=1}^{n} (U_{x_0,y_i} \times V_{x_0,y_i}).$$ If we let $U_{x_0} := \bigcap_{i=1}^{n} U_{x_0,y_i}$

then U_{x_0} is a nbhd of x_0 and a <u>finite</u> subcolln of $\mathcal{U}$ covers $U_{x_0} \times Y$. Now $\{U_x : x \in X\}$ is an open cover of X; b/c X is cpct $\exists$ $x_1, \dots, x_m \in X$ st $X = U_{x_1} \cup \dots \cup U_{x_m}$.

But then $X \times Y = (U_{x_1} \times Y) \cup \dots \cup (U_{x_m} \times Y)$. Since a finite subcolln of $\mathcal{U}$ covers each of $U_{x_i} \times Y$ $(i = 1, \dots, n)$, a finite subcolln of $\mathcal{U}$ covers $X \times Y$. $\blacksquare$

By defn of c, $\nexists$ $c' \in \mathcal{U}$ st $c' > c$. But then c has an immediate successor $c_1 \in [a,b]$.

So $[a, c_1]$ is covered by a finite sub colln of $\mathcal{U}$. $\divideontimes$

<u>CASE III</u> $c = b$.

The same reasoning as in <u>CASE II</u> shows $[a,c] = [a,b]$ is covered by a finite subcoll'n of $\mathcal{U}$. $\blacksquare$

<u>Applications</u> $[0,1] \subset \mathbb{R}$ is compact ; $[0,1]^n \subset \mathbb{R}^n$ is cpct.

<u>Heine-Borel Thm</u> Let $X \subset \mathbb{R}^n$.

Then X is cpct $\iff$ X is closed and bounded.

<u>Pf</u> $\Longrightarrow$ Suppose X is cpct. X is closed b/c $\mathbb{R}^n$ is Hausd. $X \subset \bigcup_{n \geq 1} B(\vec{0}, n)$, so $\{B(\vec{0}, n) : n = 1, 2, \ldots\}$ is an open cover of X & $\exists$ N st $X \subset B(\vec{0}, N)$ & X is bdd.

$\Longleftarrow$ Suppose X is closed & bdd. $\exists$ N st $X \subset [-N, N]^n$, so X is a closed subspace of the cpct space $[-N,N]^n$. Thus X is cpct. $\blacksquare$

Rmk If $\{X_\alpha\}$ is a family of spaces, the Tychonoff Theorem says $\prod_\alpha X_\alpha$ is cpct in the product topology.

§27 Compact subspaces of $\mathbb{R}$

Theorem Let X be an ordered set with the least upper bound property; let $a<b$ in X. Then $[a,b]$ is compact.

Pf

Let $\mathcal{U}$ be an open cover of $[a,b]$. Let

$$S = \{x \in [a,b] : \text{a finite subcolln of } \mathcal{U} \text{ covers } [a,x]\}.$$

Then $a \in S$ so $S \neq \emptyset$. Let $c = \sup S$.

CASE I $c = a$.

Choose $U \in \mathcal{U}$ st $a \in U$. If a does not have an immediate successor, $\exists\ a' \in U$ st $[a,a'] \subset U$ & $a = c \neq \sup S$. ✱ So a must have an immediate successor a_0. Then $\exists\ U' \in \mathcal{U}$ st $a_0 \in U'$, so $\{U, U'\}$ covers $[a,a_0]$. ✱

CASE II $a < c < b$.

Choose $U \in \mathcal{U}$ st $c \in U$. We claim that $[a,c]$ can be covered by a finite subcolln of $\mathcal{U}$. Indeed, if $\exists\ c' \in U$ st $c' < c$, then $[a,c']$ is covered by a finite subcolln of $\mathcal{U}$, so $[a,c]$ is too. Else, c has an immediate predecessor c_0 in $[a,b]$. So $[a,c_0]$ is covered by a finite subcolln of $\mathcal{U}$, as is $[a,c]$.

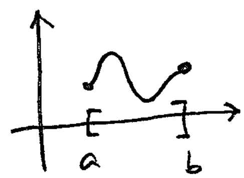
Math 190

Lecture 18

LAST TIME **Heine-Borel** $X \subset \mathbb{R}^n$ is compact $\iff$ X is closed and bounded.

- X, Y cpct $\Rightarrow$ $X \times Y$ cpct
- $f: X \to Y$ cts, surjective, X cpct $\Rightarrow$ Y cpct

Extreme Value Theorem Let $f: X \to Y$ be cts with X cpct & Y an order topology. $\exists$ $c, d \in X$ st $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.



Pf Since X is cpct & f is cts, $f(X) \subseteq Y$ is cpct.

If $f(X)$ does not have a largest elt, then

$\mathcal{U} := \{ (-\infty, f(x)) : x \in X \}$ is an open cover of $f(X)$.

B/c $f(X)$ is cpct, $\exists$ $x_1, \ldots, x_n \in X$ st $f(X) \subseteq \bigcup_{i=1}^{n} (-\infty, f(x_i))$

$$= (-\infty, \max_i f(x_i)).$$

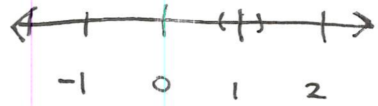
But $f(X)$ has no largest elt $\times$

This forces $f(X)$ to have a largest elt. Similarly $f(X)$ has a smallest elt. ▨

§28 Limit Point Compactness

Def A space X is limit point compact if every infinite subset $A \subset X$ has a limit point.

Ex ① $\mathbb{R}$ is not l.p.c. b/c $\mathbb{Z} \subset \mathbb{R}$ is infinite but has no limit pt.



② $X = [0,2] \cap \mathbb{Q}$ is not l.p.c. Consider a sequence

$x_1 = 1, \quad x_2 = 1.4, \quad x_3 = 1.41, \quad x_4 = 1.412 \quad \ldots \quad (\sqrt{2})$

Then $\{x_1, x_2, x_3, \ldots\}$ does not have a limit point in X.

FACT X compact $\Rightarrow$ X limit pt cpct.

Pf ~~Suppose not~~. Let $A \subset X$ be infinite with $A' = \emptyset$.

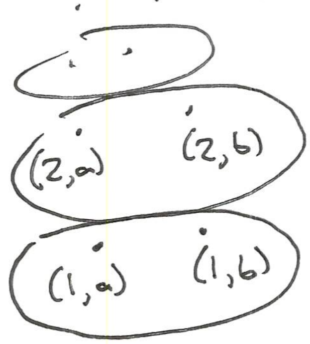
Then $\overline{A} = A \cup A' = A$, so A is closed & $X - A$ is open. For all $a \in A$, $\exists\ U_a \subset X$ open st $U_a \cap A = \{a\}$.

Now $\mathcal{U} = \{U_a : a \in A\} \cup \{X - A\}$ is an open cover of X with no finite subcover. So X is not cpct. $\boxed{\#}$

Rmk ① Let $X = \{a, b\}$ w/ indisc. top.

$Y = \mathbb{Z}_{>0} \times \{a, b\}$

Y



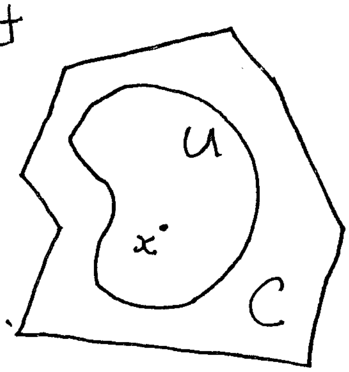
Y is limit point compact, but not compact.

② If X is a metric space,

X is cpct $\Leftrightarrow$ X is limit pt compact $\Leftrightarrow$ Every sequence in X has a convergent subsq.

§29. Local Compactness

Def A space X is locally compact if for all $x \in X$, $\exists$ a nbhd U of x & a cpt subspace $C \subset X$ st $U \subset C$.



Ex · $\mathbb{R}$ is locally cpt. $x \in (x-1, x+1) \subset [x-1, x+1]$.

· X cpt $\Rightarrow$ X locally cpt.

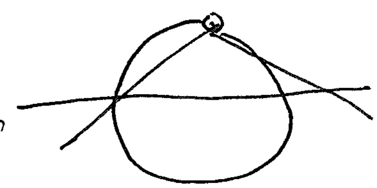
· $\mathbb{Q}$ is NOT locally cpt. (HW!)

· X discrete $\Rightarrow$ X locally cpt $(x \in \{x\})$.

* "Compact Hausdorff spaces are "nice"

"Compactification" $\mathbb{R} \hookrightarrow S^1$

locally cpt + Hausdorff ; cpt + Hausdorff



Theorem Let X be a locally compact Hausdorff space. $\exists$ a space Y s.t.

① $X \subset Y$ is a subspace of Y.

② $Y - X$ is a single pt

③ Y is compact and Hausdorff.

Furthermore, if Y' satisfies ①-③, there is a homeom. $f: Y \to Y'$ st $f(x) = x$ for all $x \in X$.

Rmk Y is the one-point compactification of X.

Pf Uniqueness Write $Y = X \cup \{\infty\}$, $Y' = X \cup \{\infty'\}$ & define $f: Y \to Y'$ by $f(x) = x$ $\forall x \in X$ and $f(\infty) = \infty'$. Then f is a bijection. Let $\mathcal{U} \subset Y$ be open. ETS $f(\mathcal{U}) \subset Y'$ is open. If $\mathcal{U} \subset X$, $f(\mathcal{U}) = \mathcal{U}$ is open in Y' b/c X is open in Y' & $X \underset{\text{subsp.}}{\subset} Y'$.

If $\mathcal{U} \not\subset X$ then $C := Y - \mathcal{U}$ is closed in Y, so is compact in X & $f(C) = C \subset Y'$ is also cpct (since $X \underset{\text{subsp.}}{\subset} Y'$).

Since Y' is Hausd., $f(C) = C$ is closed in Y', so $f(\mathcal{U}) = Y' - C$ is open in Y'. //

Existence Let $Y := X \cup \{\infty\}$ where $\infty \notin X$. Define a topology on Y by $\mathcal{U} \subset Y$ open $\Leftrightarrow$ $\mathcal{U} \subset X$ & $\mathcal{U}$ open in X, or $\infty \in \mathcal{U}$ and $Y - \mathcal{U}$ cpct in X.

CLAIM This is a topology on Y.

(i) $Y - Y = \emptyset$ is cpct so Y is open in Y

$\emptyset \subset X$ is open in X, so $\emptyset$ is open in Y.

(2) Suppose $\mathcal{U}, V \subset Y$ are open.

- If $\mathcal{U}, V \subset X$, $\mathcal{U} \cap V \subset X$ is open so $\mathcal{U} \cap V$ open in Y.
- If $\mathcal{U} \subset X$, $\infty \in V$, $\mathcal{U} \cap V = \mathcal{U} \cap (X - C)$ where $C = Y - V$ is cpct in X. B/c X is Hausd., $X - C$ is open in X, so $\mathcal{U} \cap V$ is open in X & $\mathcal{U} \cap V$ is open in Y.
- If $\infty \in \mathcal{U}$, $\infty \in V$, $Y - (\mathcal{U} \cap V) = (Y - \mathcal{U}) \cup (Y - V)$ is cpct in X (b/c both $Y - \mathcal{U}$, $Y - V$ are) so $\mathcal{U} \cap V$ is open in Y.

③ Suppose $\mathcal{U}_\alpha \subset Y$ is <u>open</u> for all $\alpha \in I$.

- If $\mathcal{U}_\alpha \subset X$ for all α, $\bigcup_\alpha \mathcal{U}_\alpha \subset X$ is open in X, hence in Y.
- If $\infty \in \mathcal{U}_\alpha$ for all α, $\left[Y - \bigcup_\alpha \mathcal{U}_\alpha\right] = \bigcap_\alpha \left[Y - \mathcal{U}_\alpha\right]$.

  $Y - \mathcal{U}_\alpha$ is cpct in X $\Leftrightarrow$ closed (b/c X Hausd.)

  So $\bigcap_\alpha Y - \mathcal{U}_\alpha$ is a closed subset of a cpct space (b/c $Y - \mathcal{U}_{\alpha_0}$ is cpct for fixed $\alpha_0 \in I$)

  So $\bigcap_\alpha \left[Y - \mathcal{U}_\alpha\right]$ is <u>cpct</u> & $\bigcup_\alpha \mathcal{U}_\alpha$ is open in X.
- If $\infty \in \mathcal{U}_\alpha$ for all $\alpha \in J$ $(J \neq \emptyset)$ and $\infty \notin \mathcal{U}_\beta$ $\forall$ $\beta \in K$ $(K \neq \emptyset)$

  w/ $I = J \cup K$, then

$$Y - \left[\bigcup_{\alpha \in J} \mathcal{U}_\alpha \cup \underbrace{\bigcup_{\beta \in K} \mathcal{U}_\beta}_{\mathcal{U},\ \text{open in } X}\right] = Y - \left[\bigcup_{\alpha \in J} \mathcal{U}_\alpha \cup \mathcal{U}\right]$$

$$= \underbrace{\left[Y - \bigcup_{\alpha \in J} \mathcal{U}_\alpha\right]}_{\text{cpct, closed in } X} - \overset{\text{open in } X}{\mathcal{U}}.$$

This is a closed subset of a cpct set, hence cpct.

So $\bigcup_\alpha \mathcal{U}_\alpha \cup \bigcup_\beta \mathcal{U}_\beta$ is open in Y. //

<u>CLAIM</u> X is a <u>subspace</u> of Y.

- If $\mathcal{U} \subset X$ is open in X then $\mathcal{U}$ is open in Y (by defn).
- If $\mathcal{U} \subset Y$ is open then $\mathcal{U} \subset X$ is open in X, or $\infty \in \mathcal{U}$ & $C = Y - \mathcal{U}$ is cpct in X, thus closed in X, so $X - C = \mathcal{U} \cap X$ is <u>open</u> in X. //

<u>CLAIM</u> Y is <u>Hausdorff</u>.

Let $x \neq y \in Y$. If $x, y \in X$, since X is Hausd $\exists$ $U, V \subset X$ open st $x \in U$, $y \in V$, $U \cap V = \emptyset$. U, V are also open in Y.

If $x \in X$, $y = \infty$, b/c <u>X is locally cpct</u> $\exists$ a nbhd $U \subseteq X$ of $\cancel{x}$ x st $U \subset C$ for some cpct st $C \subset X$.

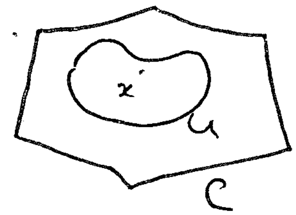
Now $Y - C$ is open, $\infty \in Y - C$, & $(Y - C) \cap U = \emptyset$.

<u>CLAIM</u> Y is <u>Compact</u>.

Let $\mathcal{U}$ be an open cover of Y. $\exists$ $U \in \mathcal{U}$ st $\infty \in U$, so $Y - U =: C$ is <u>cpct</u> ~~&~~ & $C \subseteq X$. Since $\mathcal{U}$ is an o.c. of C, $\exists$ $U_1, -, U_n \in \mathcal{U}$ st $C \subset (U_1 \cup - \cup U_n)$. So $\{U, U_1, -, U_n\}$ covers Y. $\blacksquare$

Math 190 Lecture 19 2/28/2018

Last Time • X is locally cpct if for all $x \in X$, $\exists$ a nbhd U of x & $C \subset X$ cpct st $U \subset C$.



Thm Let X be locally cpct & Hausdorff. $\exists$ a space Y st

1. $X \subset Y$ is a subspace of Y.
2. $Y - X$ is a single point
3. Y is cpct Hausdorff.

If Y' is another space satisfying 1 - 3, $\exists$ a homeo. $f: Y \to Y'$ st $f(x) = x$ for all $x \in X$.

"One-point compactification"

Pf $Y = X \cup \{\infty\}$. Open sets: (1) $U \subset X$ open, (2) $U \subset Y$ st $\infty \in U$ & $C := Y - U$ cpct in X.

LAST TIME This is a topology on Y.

CLAIM X is a subspace of Y.

- ~~Pf~~ Let $U \subset X$. If U is open in X, then U is open in Y. If U is open in Y but not in X, then $X - U =: C$ is cpct in X. B/c X is Hausd., C is closed in X so U is open in X. *

CLAIM Y is Hausdorff.

Pf Let $x \neq y \in Y$. If $x, y \in X$, $\exists$ nbhds U, V of x, y in X (hence in Y) st $U \cap V = \emptyset$. So may assume $x \in X$, $y = \infty$. Let U be a nbhd of x st $U \subset C \subset X$ for C cpct. Then $V := Y - C$ is a nbhd of ∞ & $U \cap V = \emptyset$. //

CLAIM Y ~~is~~ compact.

Pf Let $\mathcal{U}$ be an open cover of Y. $\exists$ $U \in \mathcal{U}$ st $\infty \in U$, so $C := Y - U$ is compact in X. Now $\mathcal{U}$ is an open cover of C, so $\exists$ $U_1, \dots, U_n \in \mathcal{U}$ st $C \subset (U_1 \cup \dots \cup U_n)$. Now $\{U, U_1, \dots, U_n\}$ is a finite subcover of Y. $\blacksquare$

Uniqueness Suppose $Y' = X \cup \{\infty'\}$ satisfies ① - ③.

Define $f : Y \to Y'$ by $f(x) = x$ for all $x \in X$ and $f(\infty) = \infty'$.

Let $U \subset Y$ be open. ETS $f(U) \subset Y'$ is open.

If $U \subset X$, $f(U) = U$ is open b/c X is a subsp of Y'.

Otherwise, $C := Y - U$ is cpct in Y (closed subsp. of a cpct space) & so $f(C) = C$ is cpct in Y'. B/c Y' is Hausd, C is closed in Y'. So $f(U) = Y' - C$ is open in Y'. $\blacksquare$

Groups

Def A group is a set G with a binary operation

$\star : G \times G \to G$ st

① $\exists$ $e \in G$ st $e \star g = g \star e = g$ for all $g \in G$.

② For all $x \in G$, $\exists$ $y \in G$ st $x \star y = y \star x = e$.

③ $\forall$ $x, y, z \in G$, $(x \star y) \star z = x \star (y \star z)$.

Ex ① $\mathbb{Z}$ is a gp where $\star = +$

- $0 + x = x + 0 = x \quad \forall\, x \in \mathbb{Z} \quad \Rightarrow e = 0$
- $x + (-x) = 0 \quad \forall\, x \in \mathbb{Z}$
- $(x+y)+z = x+(y+z) \quad \forall\, x,y,z \in \mathbb{Z} \qquad$ (also $x+y = y+x \ \forall\, x,y \in \mathbb{Z}$)

② $GL_n(\mathbb{R})$ is a gp $\quad A \star B = AB$ (matrix mult.)

$A I_n = I_n A = A, \quad A \cdot A^{-1} = A^{-1} \cdot A = I_n, \quad (AB)C = A(BC).$

$\Rightarrow e = I_n$

③ $\mathbb{Z}$ is NOT a gp where $\star =$ mult.

$3^{-1} \notin \mathbb{Z}.$

Rmk Given a gp G, if $\overset{x \in G}{xy = yx = e}$ & $xy' = y'x = e$,

then $y = y'$. $\quad [\, y' = y'e = y'(xy) = (y'x)y = ey = y \,]$

So write $y = x^{-1}$.

Ex The infinite cyclic gp is $\langle x \rangle = \{ \ldots, x^{-2}, x^{-1}, 1, x^1, x^2, \ldots \}$

$x^i \cdot x^j := x^{i+j}$.

$$\underset{\langle x \rangle}{x^n \in} \rightsquigarrow n \in \mathbb{Z}$$

eg $\quad x^7 x^{-3} x^2 = x^4 x^2 = x^6.$

Def Let G, H be groups. A homomorphism is a map $\varphi : G \to H$ s.t. $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G$.

A bijective homomorphism φ is an isomorphism $\ulcorner \varphi^{-1}$ is automatically a homom. $\lrcorner$

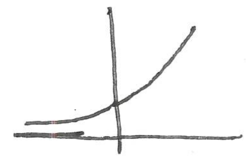
Ex ① $G = GL_n(\mathbb{R}), \quad H = \mathbb{R}^\times = \mathbb{R} - \{0\}$ (mult.)

$$\varphi : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^\times$$
$$A \longmapsto \det A$$

$\varphi(AB) = \det(AB) = \det A \cdot \det B = \varphi(A)\varphi(B) \Rightarrow \varphi$ is a homom.

(2) $G = (\mathbb{R}, +)$ $\quad H = (\mathbb{R}_{>0}, \cdot)$

$$\varphi : G \longrightarrow H \qquad (e^{x+y} = e^x e^y)$$
$$x \longmapsto e^x$$



is an <u>isomorphism</u>.

(3) $$\langle x \rangle = \{ \ldots, x^{-2}, x^{-1}, 1, x, x^2, \ldots \} \longleftrightarrow \mathbb{Z}$$
$$x^n \longleftrightarrow n$$

is an isom.

<u>Ex</u> <u>Free group on x, y</u> has elements like...

$x^2 y^{-7} x^{-3}$, x^5, 1,

$y^4 x^3 y^{-1}$, ...

<u>multiplication</u>

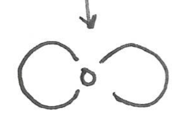
$$(x^5 y^{-4} \cancel{x^3})(\cancel{x^{-3}} y x^7 y^2)$$

$$= x^5 y^{-3} x^7 y^2.$$

~~Part~~ ~~4~~ Back to topology...

<u>Q</u> Are $\mathbb{R}$, S^1 homeo? No! $\mathbb{R}$ not cpct, S^1 cpct

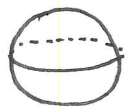
cut point!



<u>Q</u> Are $X =$ , $Y = S^1$ homeo? No.

S^1 has no cut pts

<u>Q</u> Are $X = S^2$ $\quad Y = S^1 \times S^1$ homeo?

No... but how to discern?

donut

vs??

pretzel

Math 190 | Lecture ~~##~~ 20 | 3/2/2018

Last Time Groups G = set w/ map $G\times G \to G$, $(x,y)\mapsto xy$ st

① $\forall\ x,y,z\in G$, $(xy)z = x(yz)$

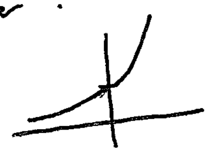
② $\exists\ e\in G$ st $xe = x = ex$ $\forall\ x\in G$,

③ $\forall\ x\in G$, $\exists\ x^{-1}\in G$ st $xx^{-1} = e = x^{-1}x$.

Ex $(\mathbb{Z},+)$, $(GL_n(\mathbb{R}), \text{matrix mult.})$, $(\mathbb{R}-\{0\}, \text{mult.})$

$\varphi: G\to H$ is a homomorphism if $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$ for all $g_1,g_2\in G$.

A bijective homomorphism is an isomorphism.

Ex $\varphi: \underset{\text{under }+}{\mathbb{R}} \longrightarrow \underset{\text{under }\cdot}{\mathbb{R}_{>0}}$, $x \longmapsto e^x$ is an isom.  $(e^{x+y} = e^xe^y)$

Ex The free group on $\{x,y\}$ has elements like...

- $xy^{-7}x^2y^3x^{-1}$ $(\neq y^{-7}x^3y^3x^{-1}$ etc.)
- $x^4y^{-1}x^{-1}$, etc.

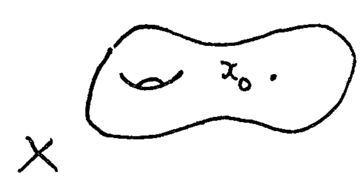
multiplication: $(x^3y^{-2}\cancel{x^3}\cancel{y})\cdot(\cancel{y^{-1}}\cancel{x^{-3}}y^4x^2y^2)$

$= x^3y^{-2}y^4x^2y^2 = x^3y^2x^2y^2.$

Similarly have free group on $\{x,y,z\}$ or any set S.

Q How to go from $(X, x_0) \rightsquigarrow \pi_1(X,x_0)$?

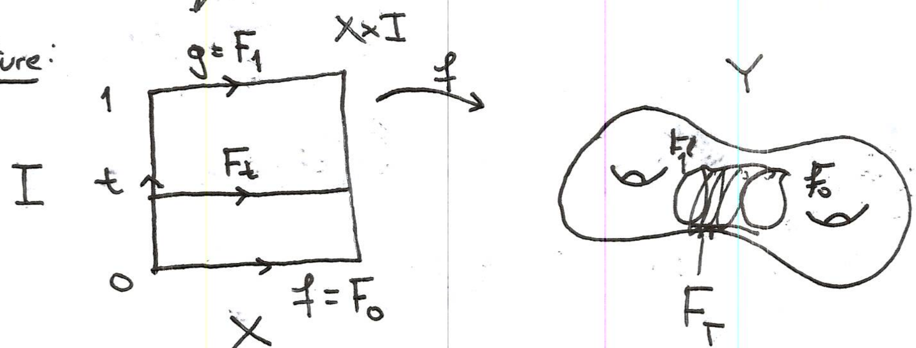
space ↑ $x_0 \in X$ group



§ 51. Homotopy.

Def Let X, Y be spaces and let $f, g : X \to Y$ be continuous fcns. f and g are homotopic ($f \simeq g$) if $\exists$ a CTS fcn $F : X \times I \to Y$ ($I = [0,1]$) s.t. $F(x,0) = f(x)$, $F(x,1) = g(x)$ for all $x \in X$.

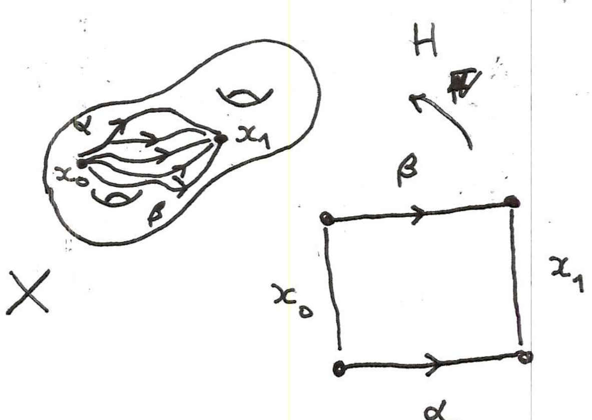
Picture:



$F_t(x) = F(x,t)$

Rmk Homotopy is an equivalence rln. (Pasting Lemma!)

Def Let $\alpha : I \to X$ and $\beta : I \to X$ be paths in X st $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1) = x_1$. α and β are path homotopic ($\alpha \simeq_p \beta$) if $\exists$ a CTS fcn $H : I \times I \to X$ s.t.

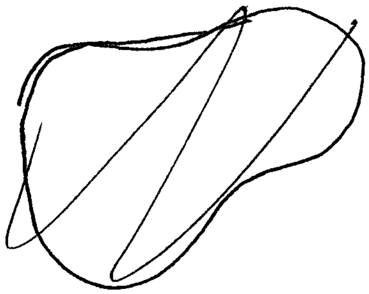


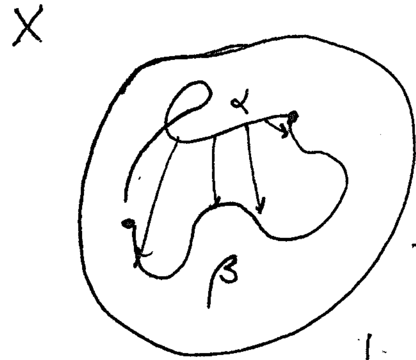
- $H(t,0) = \alpha(t)$
- $H(t,1) = \beta(t)$
- $H(0,s) = x_0$
- $H(1,s) = x_1$

for all $0 \le s, t \le 1$.

Rmk $\simeq_p$ is an equivalence rln.

Ex Let $X \subset \mathbb{R}^n$ be <u>convex</u>. (If $x, y \in X$, line segment $\overline{xy}$ is in X.)





CLAIM If α, β are paths from x_0 to x_1 in X, then α, β are path homotopic.

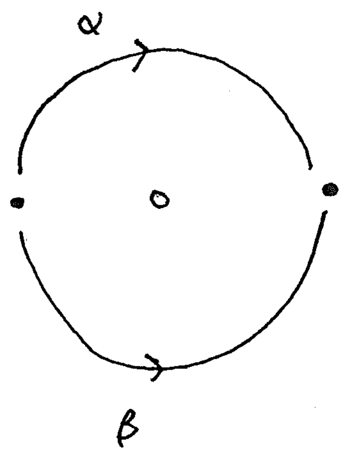
Pf Define $H : I \times I \to X$ by

$$H(t,s) = (1-s)\cdot\alpha(t) + s\cdot\beta(t)$$

for all $0 \leq s, t \leq 1$. Then H is a path homotopy btwn α and β. ▨

"Straight-line homotopy."

Ex Let $X = \mathbb{R}^2 - \{0\}$



$$\alpha : I \longrightarrow \mathbb{R}^2$$
$$t \longmapsto e^{(1-t)i\pi}$$

$$\beta : I \longrightarrow \mathbb{R}^2$$
$$t \longmapsto e^{(1+t)i\pi}.$$

Then $\alpha \not\simeq_p \beta$? (How to <u>prove</u>?)