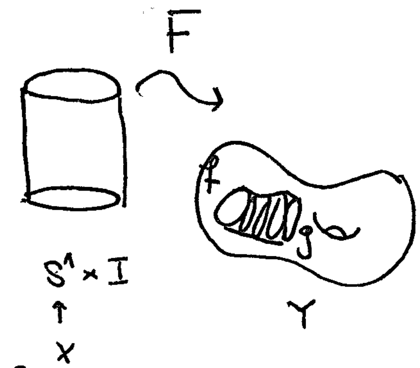
Math 190 | Lec. 21 | 3/5/18

Def– Two cts fcns $f, g : X \to Y$ are <u>homotopic</u> ($f \simeq g$) if $\exists$ a cts fcn $F : X \times I \to Y$ st

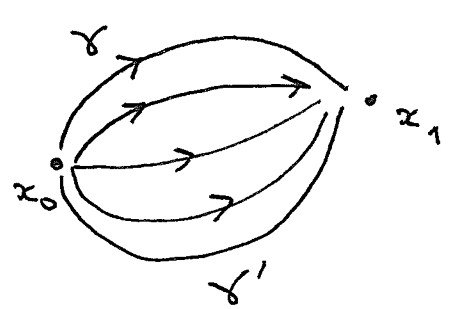
$$\begin{cases} F(x,0) = f(x) \\ F(x,1) = g(x) \end{cases} \quad \forall \; x \in X.$$



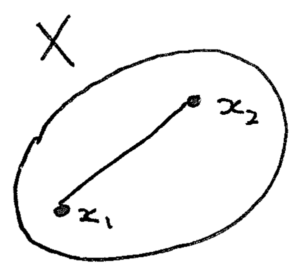
Rmk $\simeq$ is an <u>equivalence</u> rln.

Def Two <u>paths</u> $\gamma, \gamma' : I \to X$ from $x_0 \to x_1$ are <u>path</u> <u>homotopic</u> ($\gamma \simeq_p \gamma'$) if $\exists$ $F : I \times I \xrightarrow[\text{cts}]{} X$

st $F(s,0) = \gamma(s)$ $\quad F(0,t) = x_0$ $\quad$ for all $0 \le s,t \le 1$.
$F(s,1) = \gamma'(s)$ $\quad F(1,t) = x_1$



X



⌜ If $x_1, x_2 \in X$. $\overline{x_1 x_2} \subset X$ ⌟

Ex Suppose $X \subset \mathbb{R}^n$ is <u>convex</u>.

If $\gamma, \gamma' : I \to X$ are paths $x_0 \rightsquigarrow x_1$

then $\gamma \simeq_p \gamma'$.

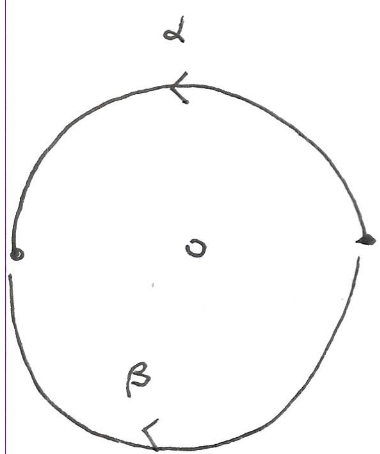
Pf Define $F : I \times I \to X$
$(s,t) \longmapsto (1-t)\gamma(s) + t\gamma'(s)$.

Then F is a path <u>homotopy</u> $\gamma \rightsquigarrow \gamma'$. "Straight-line homotopy"

However, if $X = \mathbb{R}^2 - \{0\}$.

$\alpha(s) = (\cos \pi s, \sin \pi s)$

$\beta(s) = (\cos \pi s, -\sin \pi s)$



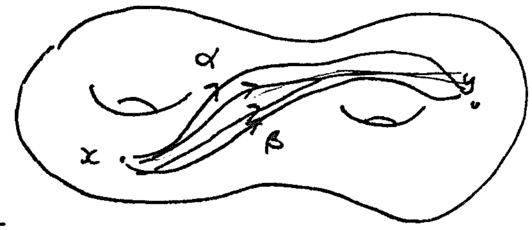
Then $\alpha \not\simeq_p \beta$.

(Hard to prove!)

Math 190 Lecture 22 3/7/2018

Last Time X - space

$\alpha, \beta : I \to X$ paths $x \rightsquigarrow y$



$\alpha \simeq_p \beta$ iff $\exists$ $F : I \times I \to X$ cts st

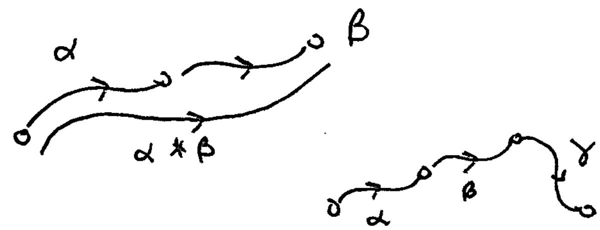
(path homotopic)

- $F(0,t) = x$ · $F(s,0) = \alpha(s)$
- $F(1,t) = y$ · $F(s,1) = \beta(s)$ $\forall$ $0 \leq s,t \leq 1$.

Q Why is this useful? * Let $[\alpha] = \{ \beta : \underset{\text{cts}}{I \to X} : \beta \underset{p}{\simeq} \alpha \}$.

Recall If $\alpha : I \to X$, $\beta : I \to X$ are paths w/ $\alpha(1) = \beta(0)$,

$$\alpha * \beta : I \to X \quad (\alpha * \beta)(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$



$\alpha(1) = \beta(0)$

$\beta(1) = \gamma(0)$

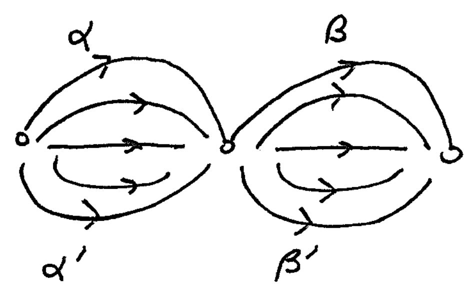
Q If $\alpha, \beta, \gamma : I \to X$ are CTS, do we have

$$\underbrace{(\alpha * \beta)}_{4\times \text{ speed}} * \underset{\uparrow\, 2\times \text{ speed}}{\gamma} \overset{?}{=} \underset{\uparrow\, 2\times \text{ spd}}{\alpha} * \underbrace{(\beta * \gamma)}_{4\times \text{ speed}} ?$$

A No!

FACT Suppose $\alpha \underset{p}{\simeq} \alpha'$ and $\beta \underset{p}{\simeq} \beta'$ with $\alpha(1) = \beta(0)$.

Then $\alpha * \beta \underset{p}{\simeq} \alpha' * \beta'$.



If $F : I \times I \to X$ is a path hty $\alpha \to \alpha'$; $G : I \times I \to X$ is a path hty $\beta \to \beta'$

then $\forall$ $H : I \times I \to X$,

$$H(s,t) = \begin{cases} F(2s,t) & 0 \leq s \leq \frac{1}{2} \\ F(2s-1,t) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

is a path hty $\alpha * \beta \to \alpha' * \beta'$.

$\Rightarrow$ We can <u>define</u> $[\alpha] * [\beta] = [\alpha * \beta]$

whenever $\alpha(1) = \beta(0)$.

<u>FACT</u> ① If $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$ for paths $\alpha, \beta, \gamma : I \to X$ then $([\alpha] * [\beta]) * [\gamma]$

$= [\alpha] * ([\beta] * [\gamma])$.

For $x \in X$, let

② ~~If~~ $e_x : I \to X$, $t \mapsto x$. If $\alpha(0) = x$ then

$$[e_x] * [\alpha] = [\alpha].$$

If $\beta(1) = x$ then $[\beta] * [e_x] = [\beta]$.

<u>Pf</u> ① We show $(\alpha * \beta) * \gamma \underset{p}{\simeq} \alpha * (\beta * \gamma)$. Indeed, define

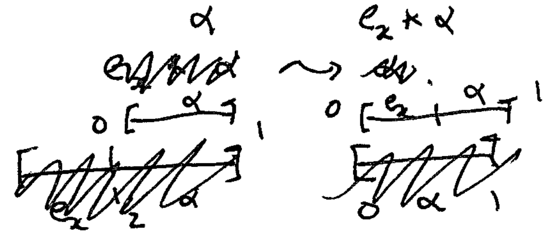
$\leftarrow [\underset{0}{\alpha} \mid_{\frac{1}{4}} \beta \mid_{\frac{1}{4}} \gamma]_1$ $\quad$ $[_0 \alpha \mid_{\frac{1}{2}} \beta \mid_{\frac{3}{4}} \gamma]_1$

$F : I \times I \to X$ by

$$F(s,t) = \begin{cases} \alpha\left(\left[\frac{4}{1+t}\right] s\right) & 0 \leq s \leq \frac{1}{4} + \frac{t}{4} \\ \beta\left(4\left(s - \frac{1+t}{4}\right)\right) & \frac{1}{4} + \frac{t}{4} \leq s \leq \frac{1}{2} + \frac{t}{4} \\ \gamma\left(\dfrac{s - \frac{1}{2} - \frac{t}{4}}{\frac{1}{2} - \frac{t}{4}}\right) & \frac{1}{2} + \frac{t}{4} \leq s \leq 1 \end{cases}$$

Then F is a path hty $(\alpha * \beta) * \gamma \to \alpha * (\beta * \gamma)$. //

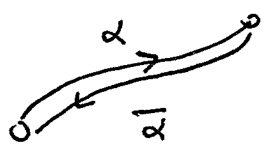
(2) We give a path hty $\alpha \rightsquigarrow e_x * \alpha$.



Define $F: I \times I \to X$ by $F(s,t) = \begin{cases} x & 0 \le s \le t/2 \\ \alpha\left(\dfrac{s - t/2}{1 - t/2}\right) & t/2 \le s \le 1 \end{cases}$.

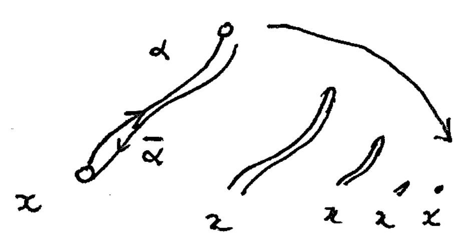
Then F is a path hty $\alpha \rightsquigarrow e_x * \alpha$. //.
Similarly $\beta * e_x \simeq_p \beta$.

Moreover, given $\alpha : I \to X$, define $\bar{\alpha} : I \to X$ by $\bar{\alpha}(t) = \alpha(1-t)$.



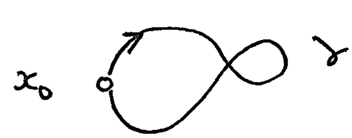
"reverse path".

<u>FACT</u> Suppose $\alpha : I \to X$ is a path w/ $\alpha(0) = x$.
Then $[\alpha * \bar{\alpha}] = [e_x]$.



($\underline{\text{Pf}}$ Define $F : I \times I \to X$ by ...)

<u>Def</u> Let $x_0 \in X$. A loop at x_0 is a path $\gamma : I \to X$ w/ $\gamma(0) = \gamma(1) = x_0$.



<u>Obs</u> If γ, γ' are loops at x_0 then $\gamma * \gamma'$, $[\gamma] * [\gamma']$ make sense!

Def Let X be a topological space & let $x_0 \in X$.

The **fundamental group** $\pi_1(X, x_0)$ is

$$\pi_1(X, x_0) = \{ [\gamma] \ ; \ \gamma \text{ a loop at } x_0 \text{ in } X \}$$

with operation $[\gamma] * [\gamma'] := [\gamma * \gamma']$.

(Assoc. ✓ Identity: $[e_{x_0}]$ ✓ Inverses $[\gamma]^{-1} = [\bar{\gamma}]$ ✓)

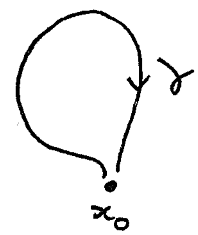
Ex

Math 190 Lecture 23 3/9/2018

Last Time - $\alpha : I \xrightarrow[\text{path}]{} X$ $\quad [\alpha] = \{ \alpha' : \alpha' \underset{p}{\simeq} \alpha \}$.

- $[\alpha] * [\beta] = [\alpha * \beta]$ $\quad (\alpha(1) = \beta(0))$

- If $x_0 \in X$, the fundamental group is



$$\pi_1(X, x_0) := \{ [\gamma] : \gamma \text{ a loop in } X \text{ at } x_0 \}$$

identity $[e_{x_0}]$ $\quad$ inverses $[\gamma]^{-1} = [\bar{\gamma}]$ (reversal)

Ex Suppose $X \subseteq \mathbb{R}^n$ is ~~compact~~ convex. Then $\pi_1(X, x_0) = 0$.

Why? Let $\gamma : I \longrightarrow X$ be a loop at x_0.

Then $F(s,t) = (1-t)\gamma(s) + t x_0$ is a path htpy from γ to

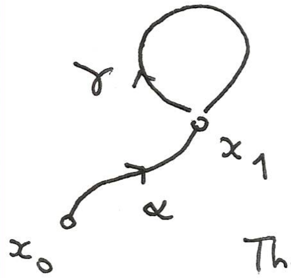
So $[\gamma] = [e_{x_0}]$ in $\pi_1(X, x_0)$

and $\pi_1(X, x_0) = \{ [e_{x_0}] \} = 0$. (1-elt gp!)

Q Does $\pi_1(X, x_0)$ depend on x_0?

FACT Let $x_0, x_1 \in X$ & suppose $\alpha : I \longrightarrow X$ is a path $x_0 \rightsquigarrow x_1$ in X. Then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ (group isom.)

⌜ Just write $\pi_1(X)$ for $\pi_1(X, x_0)$ if X is path connected. ⌟

Pf 

Define $\hat{\alpha} : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ by

$$[\gamma] \longmapsto [\alpha] * [\gamma] * [\bar{\alpha}].$$

Then $\hat{\alpha}$ is a group homom: if $[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$

then $\hat{\alpha}([\gamma_1] * [\gamma_2]) = [\alpha] * [\gamma_1] * [\gamma_2] * [\bar{\alpha}]$

$[\bar{\alpha}] * [\alpha] = [e_{x_1}]$

$$= ([\alpha] * [\gamma_1] * [\bar{\alpha}]) * ([\alpha] * [\gamma_2] * [\bar{\alpha}])$$

$$= \hat{\alpha}([\gamma_1]) * \hat{\alpha}([\gamma_2]).$$

Define $\widehat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$

$$[\beta] \longmapsto [\bar{\alpha}] * [\beta] * [\alpha].$$

Then $\widehat{\alpha}$ is a gp homom (same reasoning). If

$[\gamma] \in \pi_1(X, x_1)$,

$$\widehat{\alpha}(\alpha[\gamma]) = \widehat{\alpha}([\alpha] * [\gamma] * [\bar{\alpha}]) = [\bar{\alpha}] * [\alpha] * [\gamma] * [\bar{\alpha}] * [\alpha]$$
$$= [\gamma]$$

so $\widehat{\alpha} \circ \hat{\alpha} = \mathrm{id}_{\pi_1(X, x_0)}$. Similarly $\hat{\alpha} \circ \widehat{\alpha} = \mathrm{id}_{\pi_1(X, x_1)}$.

So $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. ▨

FACT Let $f : (X, x_0) \longrightarrow (Y, y_0)$ be cts (i.e $f : X \to Y$ cts, $f(x_0) = y_0$.)

If $\gamma : I \to X$ is a loop at x_0 in X then $f \circ \gamma : I \longrightarrow Y$

is a loop at y_0 in Y.

Def Let $f:(X,x_0) \to (Y,y_0)$ be a CTS map.

Then
$$f_\bullet : \pi_1(X,x_0) \longrightarrow \pi_1(Y,y_0)$$
$$[\gamma] \longmapsto [f \circ \gamma]$$
is the <u>homomorphism</u> <u>induced</u> <u>by</u> <u>f</u>.

⌜Use $f\circ(\alpha * \beta) = (f\circ\alpha) * (f\circ\beta)$.⌟

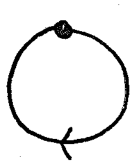
FACT ① If $\iota : (X,x_0) \to (X,x_0)$ is the identity map, then $\iota_\bullet : \pi_1(X,x_0) \to \pi_1(X,x_0)$ is the id. map.

② If $(X,x_0) \xrightarrow{f} (Y,y_0) \xrightarrow{g} (Z,z_0)$ are CTS, then $(f\circ g)_\bullet = f_\bullet \circ g_\bullet$.

☆ If X, Y are path connected spaces with $X \underset{\text{homeo.}}{\cong} Y$ then $\pi_1(X) \underset{\text{iso.}}{\cong} \pi_1(Y)$. ☆

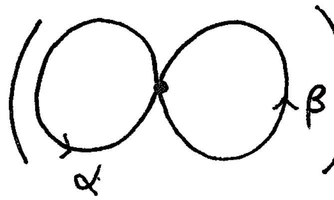
<u>Q</u> Do we <u>ever</u> have $\pi_1(X) \neq 0$?

· $\pi_1(S^1_\bullet) = \langle [\gamma] \rangle \cong \mathbb{Z}$



γ

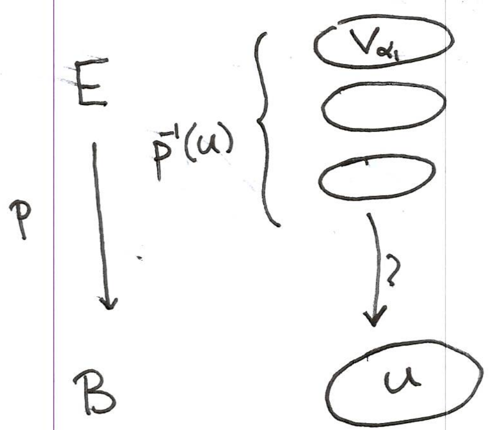
· $\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \oplus \pi_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$

· $\pi_1\Big($  $\Big) \cong$ Free group on $\{a, b\}$

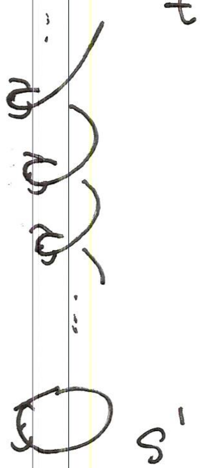
<u>Q</u> How to <u>prove</u> $\pi_1(S^1) \cong \mathbb{Z}$?

Def A CTS surjn $p: E \to B$ is a <u>covering projection</u> if $\exists$ an open cover $\mathcal{U}$ of B s.t. for every $U \in \mathcal{U}$,

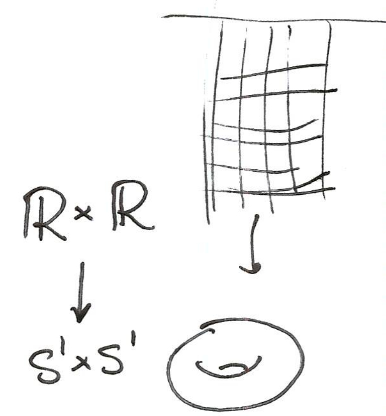
- $p^{-1}(U) = \coprod_\alpha V_\alpha$ for some opens $V_\alpha \subset E$, &
- $p|_{V_\alpha}: V_\alpha \xrightarrow{\sim} U$ is a homeom. for all α.



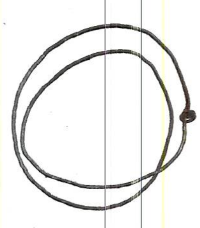
Ex $p: \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ is a covering map.



- $p: S^1 \amalg S^1 \to S^1$, $(x_1, x_2) \mapsto x$ is covering.



- $p: S^1 \to S^1$, $z \mapsto z^2$ is covering

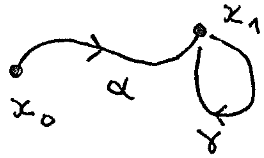


FACT If $p_1: E_1 \to B_1$, $p_2: E_2 \to B_2$ are covering, so is $p_1 \times p_2: E_1 \times E_2 \to B_1 \times B_2$.

Math 190 Lecture 24 3/12/2018

Last Time $*$ $f : (X, x_0) \longrightarrow (Y, y_0)$ CTS

$$\Rightarrow \quad f_\bullet : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0) \qquad \cdot\ id_\bullet = id$$
$$[\gamma] \longmapsto [f \circ \gamma] \qquad \cdot\ (f \circ g)_\bullet = f_\bullet \circ g_\bullet$$
(homom.)

$*$ If 

$\alpha : x_0 \rightsquigarrow x_1$ is a path,

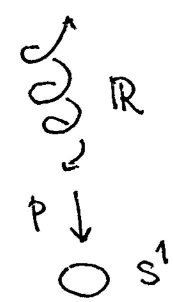
$$\hat{\alpha} : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$$
$$[\gamma] \longmapsto [\alpha * \gamma * \bar{\alpha}]$$
is an isom.

$\pi_1(X, x_0) \equiv \pi_1(X)$ if X is path connected.

GOAL $\pi_1(S^1) \cong \mathbb{Z}$

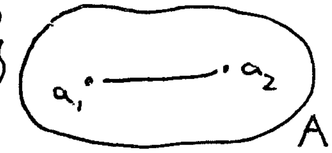
Def $p : E \longrightarrow B$ (cts surj.) is a covering map if $\exists$ an open cover $\mathcal{U}$ of B st for all $U \in \mathcal{U}$, $p^{-1}(U) = \bigcup_\alpha V_\alpha$ w/

- $V_\alpha \cap V_{\alpha'} = \emptyset$ if $\alpha \neq \alpha'$
- $V_\alpha \subset E$ open for all α
- $p|_{V_\alpha} : V_\alpha \xrightarrow{\simeq} U$ a homeomorphism $\forall \alpha$.

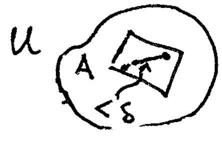


Let (X, d) be a metric space. The diameter of $A \subset X$ is

$$\operatorname{diam} A := \sup \{ d(a_1, a_2) : a_1, a_2 \in A \}$$

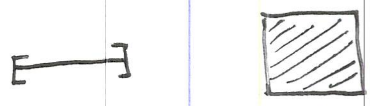


Lebesgue Number Lemma Let (X, d) be a compact metric space & let $\mathcal{U}$ be an open cover of X. $\exists\ \delta > 0$ such that for all $A \subset X$ with $\operatorname{diam} A < \delta$, $\exists$ $U \in \mathcal{U}$ st $A \subset U$.



⌜δ = "Lebesgue # for $\mathcal{U}$"⌟

* For us, $X = I$ or $I \times I$.



<u>Def</u> Let $p: E \to B$ be a covering map & let $f: X \to B$ be a CTS fcn. A <u>lift</u> of f is a CTS fcn $\tilde{f}: X \to E$ st $f = p \circ \tilde{f}$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

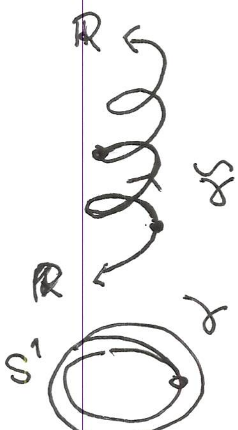
(<u>Path Lifting</u>)

<u>Theorem</u> Let $p: E \to B$ be a covering map w/ $p(e_0) = b_0$. Let $\gamma: I \to B$ be a path w/ $\gamma(0) = b_0$. $\exists!$ lift $\tilde{\gamma}: I \to E$ st $\tilde{\gamma}(0) = e_0$.

<u>Ex</u> $p: \mathbb{R} \to S^1$, $x \mapsto e^{2\pi i x}$.

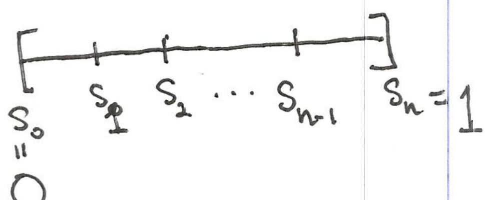
$\gamma: I \to S^1$, $s \mapsto e^{3\pi i s}$ $\qquad$ $\tilde{\gamma}: I \to \mathbb{R}$, $\tilde{\gamma}: u \mapsto \frac{3}{2}u + 2$.

$b_0 = 1$; $e_0 = 2$



<u>Pf</u> Let $\mathcal{U}$ be the open cover of B satisfying covering sp. axioms. Then $\gamma^{-1}(\mathcal{U}) := \{\gamma^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of I. Let $\delta > 0$ be a Lebesgue # of this cover & choose

$0 = s_0 < s_1 < \cdots < s_n = 1$ st $(s_i - s_{i-1}) < \delta$ for all i.



We define $\tilde{\gamma}$ <u>piece</u> by <u>piece</u>. Suppose $\tilde{\gamma}$ has been defined on $[s_0, s_1], \ldots, [s_{i-1}, s_i]$. Let $\tilde{\gamma}(s_i) = e_i$, $\gamma(s_i) = b_i$, so $p(e_i) = b_i$. By our choice of δ, $\exists$ $U \in \mathcal{U}$ st

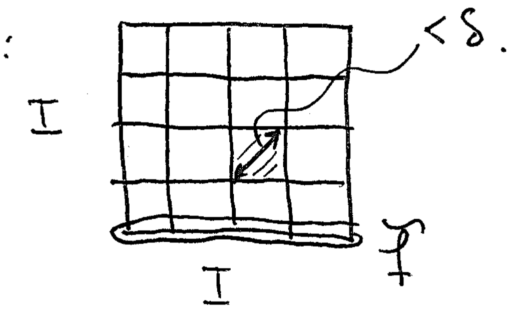
$\gamma([s_i, s_{i+1}]) \subset U$. Write $p^{-1}(U) = \coprod_\alpha V_\alpha$. $\exists!$ α_0 st $\tilde{\gamma}(s_i) = e_i \in V_{\alpha_0}$. Now for $s_i \leqslant s \leqslant s_{i+1}$ set $\tilde{\gamma}(s) := (p|_{V_{\alpha_0}})^{-1} \circ \gamma(s)$. This defines $\tilde{\gamma}$ on $[s_i, s_{i+1}]$; by induction we have $\tilde{\gamma} : I \xrightarrow[\text{CTS}]{} E$ st $\tilde{\gamma}(0) = e_0$. For uniqueness, use that $[s_i, s_{i+1}]$ is <u>connected</u> for all i, so that if $\tilde{\gamma}' : I \xrightarrow[\text{CTS}]{} E$ lifts γ then $\tilde{\gamma}'([s_i, s_{i+1}])$ meets <u>just</u> <u>one</u> V_{α_0} for each i. ▨

<u>Theorem</u> (<u>Homotopy</u> <u>Lifting</u>) Let $p : E \to B$ be a covering map and let $F : I \times I \longrightarrow B$ be CTS. Define
$$f : I \to B, \quad s \mapsto F(s,0)$$
& suppose $\exists\ \tilde{f} : I \to E$ which lifts f. $\exists!$ lift $\tilde{F} : I \times I \to E$ of F st $\tilde{F}(s,0) = \tilde{f}(s)$ for all $0 \leqslant s \leqslant 1$.

If F is a path homotopy, so is $\tilde{F}$.

<u>Pf</u> <u>Sketch</u> Let $\mathcal{U}$ be an o.c. of B satisfying covering map axioms; $F^{-1}(\mathcal{U}) = \{F^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of $I \times I$. Let $\delta > 0$ be a Lebesgue #; chop $I \times I$ into squares of diam. $< \delta$:

Now define $\tilde{F} : I \times I \to E$ "from bottom to top".



For F a path hty use that $p^{-1}\{b\}$ is <u>discrete</u> <u>for all $b \in B$</u>. ▨

<u>Theorem</u> ~~The fundamental~~ $\pi(S^1)$ is isomorphic to $\mathbb{Z}$.

<u>Pf</u> Let $x_0 = 1 \in S^1$ & define $p: \mathbb{R} \to S^1$
$x \mapsto e^{2\pi i x}$.

Then p is a covering map & $p^{-1}(x_0) = \mathbb{Z}$.

Given any <u>loop</u> $\gamma: I \to S^1$ at x_0 in S^1, we have a <u>unique</u> lift $\tilde{\gamma}: I \to \mathbb{R}$ st $\tilde{\gamma}(0) = 0$, so $p \circ \tilde{\gamma} = \gamma \Rightarrow \tilde{\gamma}(1) \in \mathbb{Z}$. Furthermore, if $\gamma \simeq_p \gamma'$ are path htic loops in S^1 at x_0, $\tilde{\gamma} \simeq_p \tilde{\gamma}'$ so $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$. <u>Define</u>

$$\varphi: \pi_1(S^1, x_0) \longrightarrow \mathbb{Z}$$
$$[\gamma] \longmapsto \tilde{\gamma}(1)\ ;$$ φ is a <u>well-defined</u> <u>fcn</u>.

If $[\gamma_1], [\gamma_2] \in \pi_1(S^1, x_0)$ then $\widetilde{\gamma_1 * \gamma_2}(1) = \tilde{\gamma}_1(1) + \tilde{\gamma}_2(1)$
<u>check!</u>

so φ is a group homom.

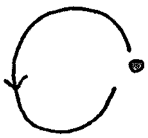
If $\varphi([\gamma_1]) = \varphi([\gamma_2])$ then $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$ (and $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) = 0$) so $\tilde{\gamma}_1 \simeq_p \tilde{\gamma}_2$ (paths in $\mathbb{R}$!) and $\gamma_1 \simeq_p \gamma_2$. $\ulcorner F: I \times I \to \mathbb{R}$ p.h.: $\tilde{\gamma}_1 \rightsquigarrow \tilde{\gamma}_2 \Rightarrow p \circ F: I \times I \to S^1$ p.h. $\gamma_1 \rightsquigarrow \gamma_2$. $\lrcorner$

So $[\gamma_1] = [\gamma_2]$ & φ is injective.

Also if $\gamma_n: I \to S^1$, $s \mapsto \exp(2\pi i n s)$ $(n \in \mathbb{Z})$ then $\tilde{\gamma}_n(1) = n$ so $\varphi: [\gamma_n] \mapsto n$ & φ is <u>surjective</u>. ▨

Math 190 Lecture 25 3/14/2018

Last Time • For $x_0 \in S^1$, $\pi_1(S^1, x_0) \cong \mathbb{Z}$
$= \langle [\gamma] \rangle$.

$\gamma(s) = x_0 e^{2\pi i x_0}$  $\bullet\, x_0$

Other Fundamental Groups

• $\pi_1(\overbrace{S^1 \times S^1 \times \cdots \times S^1}^{n}) \cong \overbrace{\pi_1(S^1) \times \cdots \times \pi_1(S^1)}^{n} \cong \mathbb{Z}^n$.

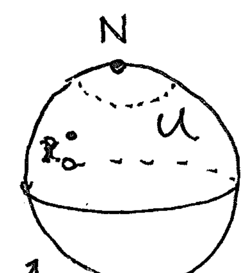
"n-torus"

$\mathbb{Z}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{Z} \right\}$

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$.

CLAIM If $n > 1$ then $\pi_1(S^n) = 0$.

Pf Let $N := (0 \ldots 0\, 1) \in S^n$ be the north pole & choose $x_0 \in S^n - \{N\}$. Let $\gamma : I \to S^n$ be a loop at x_0. If $N \notin \gamma(I)$, since $S^n - \{N\} \underset{\text{homeo}}{\cong} \mathbb{R}^n$, $\gamma \underset{p}{\simeq} e_{x_0}$ so $[\gamma] = 0$ in $\pi_1(S^n, x_0)$.



So ETS $\gamma \underset{p}{\simeq} \gamma'$ where $N \notin \gamma'(I)$.

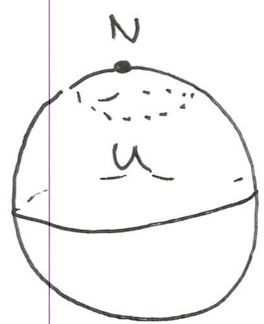
Let U be a disc arnd N in S^n not containing x_0. Then $\gamma^{-1}(V)$ is an open subset of I, so that $\gamma^{-1}(V) = \bigcup_i J_i$ is a disjoint union of open ints $J_i = (a_i, b_i)$ w/ $a_i < b_i$. Then $C := \gamma^{-1}(N)$ is closed in I, hence compact. Since $\{J_i\}$ is an open

I [———]

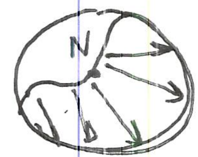
$V = \pi^{-1}(U) \subset I$

cover of C, $\exists$ finitely many i (say $i_1, \dots, i_m$) st

$C \cap J_i \neq \emptyset$. Now perform a path hty $\gamma \rightsquigarrow \gamma'$ by altering γ on $i_1, \dots, i_m$ st N is not hit:



$\rightsquigarrow$

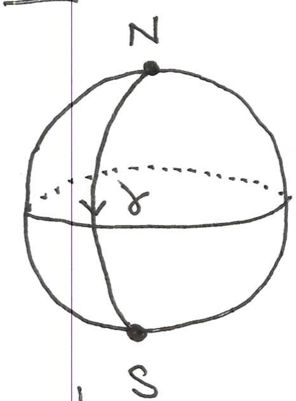


. Since this need only be done finitely many times we get a path hty $\gamma \rightsquigarrow \gamma'$ where $N \notin \gamma'(I)$. $\blacksquare$

Recall $\mathbb{RP}^n = S^n / x \sim -x$. $(\mathbb{RP}^1 \underset{\text{homeo}}{\cong} S^1)$ $\pi_1(\mathbb{RP}^1, x_0) \cong \mathbb{Z}$

Fact For $n > 1$, $\pi_1(\mathbb{RP}^n) \cong \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\{0,1\}}$

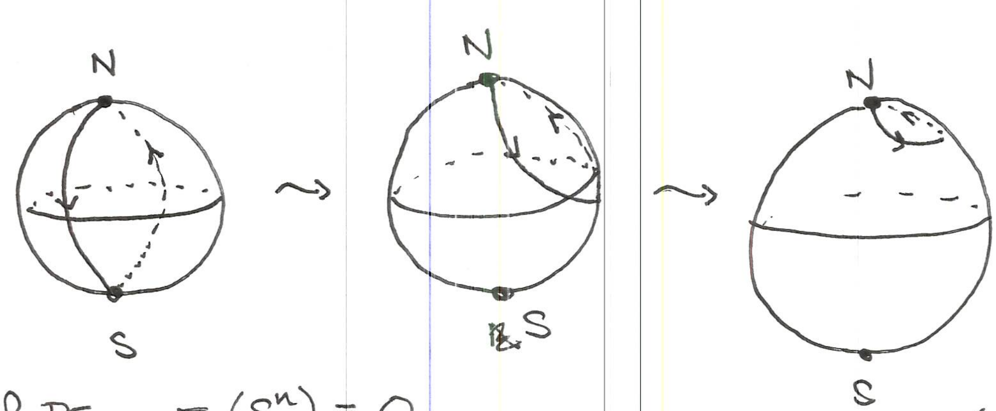
$0 + 0 = 0$
$1 + 0 = 0 + 1 = 1$
$1 + 1 = 0$.



$\pi_1(\mathbb{RP}^n, N)$

$[\gamma] * [\gamma] = [e_N]$

$\gamma * \gamma$



$\rightsquigarrow e_N$.

Idea of Pf $\cdot\ \pi_1(S^n) = 0$ $\cdot\ \pi_1(S^n/x\sim -x) \cong \mathbb{Z}/2\mathbb{Z}$

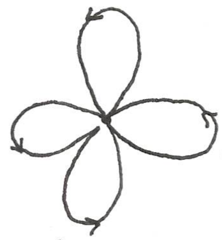
$\cdot\ \mathbb{Z}/2\mathbb{Z}$ "acts ~~properly~~ freely + properly discontinuously" on S^n via $x \leftrightarrow -x$. "p.c., l.p.c., semilocally simply conn."

Homotopy Equivalence

Recall If $\alpha : I \longrightarrow X$ is a path $x_0 \longrightarrow x_1$,

have an isomorphism

FACT



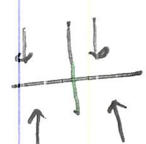
$\pi_1(\underbrace{S^1 \vee \dots \vee S^1}_{\text{bouquet of } n \text{ circles}}) \cong$ free group on $x_1, x_2, \dots, x_n$.

eg $\quad x_1^2 x_2^{-7} x_3 x_1^{-2} x_4 x_2$

Applications?

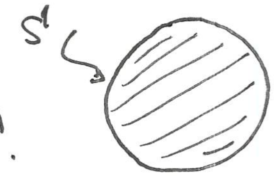
Def Let $A \subset X$. A retraction $r : X \to A$ is a CTS fcn st $r(a) = a$ for all $a \in A$.

Ex $r : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x$



is a retraction.

D^2



Theorem There is no retraction $r : D^2 \to S^1$.

$(D^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\})$

Proof If $r : D^2 \to S^1$ were a retraction, let $j : S^1 \hookrightarrow D^2$ be inclusion. Then $r \circ j = \mathrm{id}_{S^1}$, so on fundamental gps

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "j_*"] \arrow[rr, bend right, "((\mathrm{id})_{S^1})_* = \mathrm{id}"'] & \pi_1(D^2) \arrow[r, "r_*"] & \pi_1(S^1) \end{tikzcd}$$

But $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(D^2) = 0$ (D^2 is convex) so $j_* = 0$.

$\divideontimes$