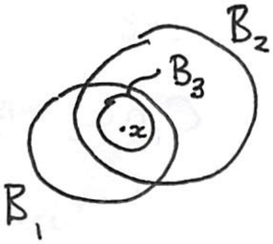
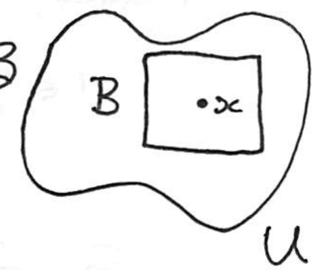
Math 190 — Lecture 3 — 1/12/2018

Last Time X - set, $\mathcal{B}$ - coll'n of subsets of X

- $\mathcal{B}$ is a basis for a topology on X if
  1. $\forall\ x \in X$, $\exists\ B \in \mathcal{B}$ st $x \in B$,
  2. $\forall\ B_1, B_2 \in \mathcal{B}$, $\forall\ x \in B_1 \cap B_2$, $\exists\ B_3 \in \mathcal{B}$ st $x \in B_3 \subset B_1 \cap B_2$



- The topology gen'd by $\mathcal{B}$ has $U \subset X$ open $\iff$ for all $x \in U$, $\exists\ B \in \mathcal{B}$ s.t. $x \in B \subset U$.



$\iff$ U is a union of sets in $\mathcal{B}$

Q Given a top. space X & a family $\mathcal{C}$ of open sets, how to tell whether $\mathcal{C}$ is a basis for the topology of X?

Fact Let X be a top. space & $\mathcal{C}$ a family of open sets in X. If

(*) for all $U \subset X$ open, for all $x \in U$ $\exists$ $C \in \mathcal{C}$ st $x \in C \subset U$

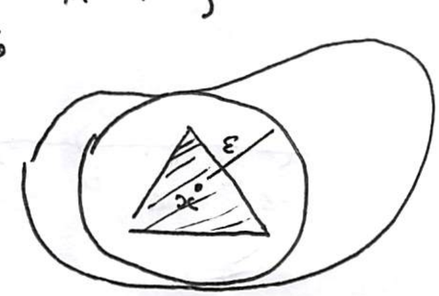
then $\mathcal{C}$ is a basis for the topology of X.

Ex $X = \mathbb{R}^2$, standard topology

$\mathcal{C} = \{$all open triangles  in $\mathbb{R}^2\}$

$\Rightarrow$ $\mathcal{C}$ is a basis for standard top. of $\mathbb{R}^2$:

$U \subset \mathbb{R}^2$ open, $x \in U$



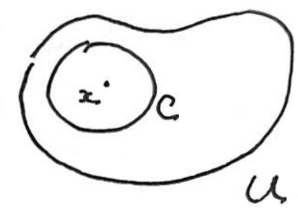
Pf We claim $\mathcal{C}$ is the basis for <u>some</u> top. $\mathcal{T}$ on X.

Indeed, if $x \in X$ $\exists$ $C \in \mathcal{C}$ st $x \in C \subset X$ so $\bigcup_{C \in \mathcal{C}} C = X$. Also if $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$, $C_1 \cap C_2$ is <u>open</u> in X so $\exists$ $C_3 \in \mathcal{C}$ st $x \in C_3 \subset (C_1 \cap C_2)$.

Now we show $\mathcal{T}$ is the given top. on X.

Indeed, for $U \subset X$ we have

$U \in \mathcal{T} \iff$ for every $x \in U$ $\exists$ $C \in \mathcal{C}$ such that $x \in C \subset U$



So for all $U \subset X$ open, we have $U \in \mathcal{T}$.

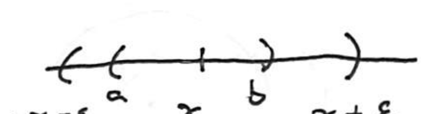
Also, $\mathcal{T} = \{\text{all unions of sets in } \mathcal{C}\}$ so that (b/c every set in $\mathcal{C}$ is <u>open</u> & unions of open sets are <u>open</u>), every set $U \in \mathcal{T}$ is open in X. //

Ex $\mathbb{R}$ – standard topology has basis $\{(a,b) : a < b \text{ in } \mathbb{R}\}$. uncountable

However, $\mathcal{B} = \{(a,b) : a, b \in \mathbb{Q}, a < b\}$ countable

is <u>also</u>! a basis for std topology of $\mathbb{R}$!

<u>Why?</u> Every set in $\mathcal{B}$ is open. Let $U \subset \mathbb{R}$ be open, $x \in U$. $\exists$ $\varepsilon > 0$ st $(x-\varepsilon, x+\varepsilon) \subset U$. Choose $a, b \in \mathbb{Q}$ st $x - \varepsilon < a < x < b < x + \varepsilon$.

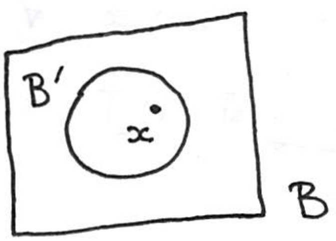


Then $x \in (a,b) \subset U$. So $\mathcal{B}$ is a basis for the top. of $\mathbb{R}$. //

Fact Let $\mathcal{B}, \mathcal{B}'$ be bases for the topologies $\mathcal{T}, \mathcal{T}'$ on a set X. TFAE:

① $\mathcal{T}' \supset \mathcal{T}$

② for all $B \in \mathcal{B}$, for all $x \in B$, $\exists\ B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.



basis: $\{(a,b) : a<b \text{ in } \mathbb{R}\}$

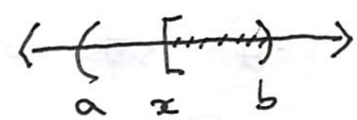
$\mathcal{S}$

$\mathbb{R}$ (std. top.)

Ex $\mathbb{R}_\ell$ is strictly finer than $\mathbb{R}$ (std. top.)

- $[0,1)$ is open in $\mathbb{R}_\ell$ but not in $\mathbb{R}$.
- If $(a,b) \subset \mathbb{R}$ for $a<b$ & $x \in (a,b)$

then $x \in [x,b) \subset (a,b)$.



So every open set in $\mathbb{R}_\ell$ is also open in $\mathbb{R}$.

Topologies on $\mathbb{R}$:

indiscrete $\subsetneq$ finite complement $\subsetneq$ standard $\subsetneq$ $\mathbb{R}_\ell$ $\subsetneq$ discrete

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§14. The Order Topology

Ordered Sets

Def An ordered set is a set X w/ a binary rln $<$ s.t.

① for all $x, y \in X$ exactly one of $x<y$, $x=y$, or $x>y$ holds.

② for all $x, y, z \in X$, if $x<y$ and $y<z$ we have $x<z$,

eg $(\mathbb{R}, <)$ usual $\qquad (\mathbb{Q}, <)$ usual $(\{1,2,3\}, <)$

$(\mathbb{Z}, <)$ usual $\qquad$ etc. $(\mathbb{Z}_{\geq 0}, <)$.

$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

<u>Def</u> Let X be an ordered set. An elt $a \in X$ is a <u>smallest elt</u> if $a \leq x \ \forall \ x \in X$.

An elt $b \in X$ is a <u>largest elt</u> if $x \leq b \ \forall \ x \in X$.

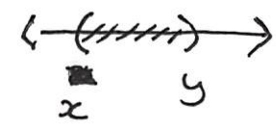
For $x, y \in X$ define

$[x,y] := \{ z \in X : x \leq z \leq y \}$, $\overbrace{[x,y), (x,y], (x,y)}^{\text{etc.}}$.

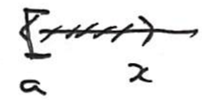
<u>Def</u> Let X be an ordered set with ≥ 2 elements.

The <u>order topology on X</u> has basis consisting of sets of the form:

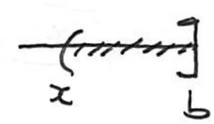
① (x,y) for $x < y$ in X



② $[a,x)$ if $a \in X$ is a smallest elt

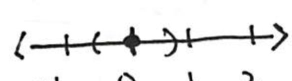


③ $(x,b]$ if $b \in X$ is a largest elt



<u>Ex</u> - Order topology on $\mathbb{R}$ = std. top. on $\mathbb{R}$

- Order top. on $\mathbb{Z}$ = discrete top on $\mathbb{Z}$



$\forall z \in \mathbb{Z}$, $(z-1, z+1) = \{z\}$ is <u>open</u>.

Def Let $(X,<)$ and $(Y,<)$ be ordered sets. The <u>dictionary order</u> on $X \times Y$ is

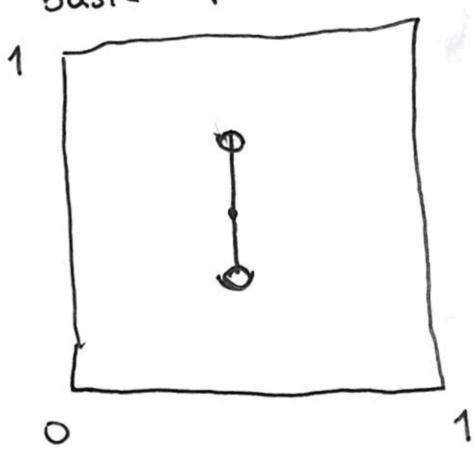
$$(x_1,y_1) < (x_2,y_2) \iff x_1 < x_2 \text{ or } (x_1 = x_2 \text{ and } y_1 < y_2)$$

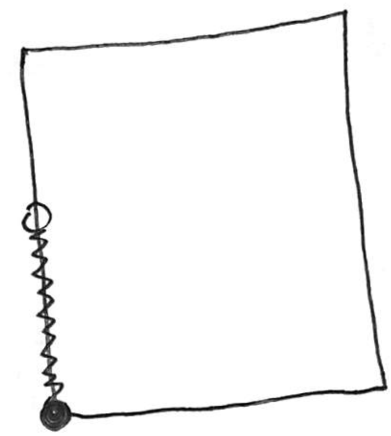
Ex $I = [0,1]$

$I \times I = [0,1] \times [0,1]$

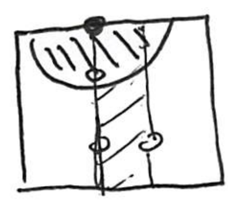
The dictionary order on $I \times I$ is the <u>ordered square</u>.

Some basic open sets:



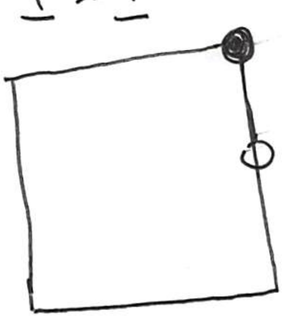


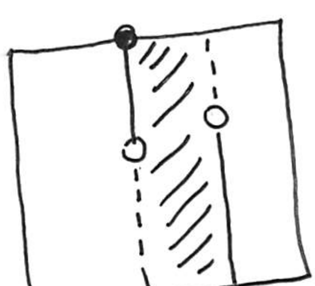
<u>Rmk</u>

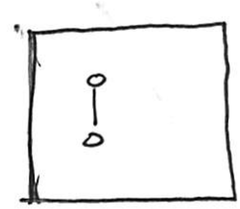


open in std, not dict order

$I \times I$







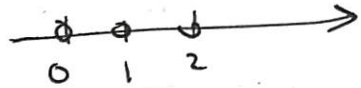
open in dict order, not standard

An order $<$ on X is a <u>well ordering</u> if for all $A \subset X$ st $A \neq \emptyset$, A has a smallest element.

(Say $(X, <)$ is <u>well ordered</u>)

eg ($<$ = usual ordering)

- $\{1, 2, 3\}$ is well ordered.
- $\mathbb{Z}_{\geq 0}$ is well ordered



- $\mathbb{Z}_{\leq 0}$ is <u>NOT</u> well ordered

- $\mathbb{Q}, \mathbb{R}$ are <u>NOT</u> well ordered

<u>Theorem</u> Let X be <u>any</u> set. There exists an order $<$ on X which is a well ordering.