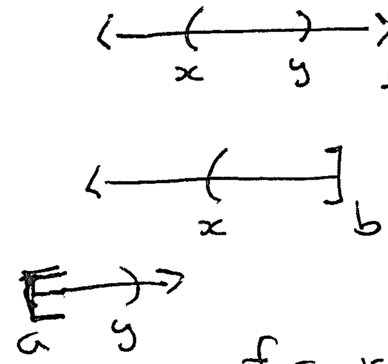
Math 190 — Lecture 4 — 1/17/2018

## Last Time

$X$ - ordered set w/ $\geq 2$ elements $\Rightarrow$ order topology

basis:




for $x<y$ in $X$

if $b$ is largest

if $a$ is smallest.

**Def** Let $<$ be an order on a set $X$.
~~Theorem~~ $<$ is a well order if for all nonempty $A \subseteq X$, $A$ has a smallest element.

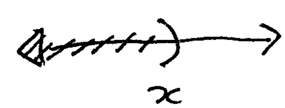
**Ex**

| $X$ | well ordered? |
|---|---|
| $\{1,2,3\}$ | ✓ |
| $\mathbb{Z}_{\geq 0}$ | ✓ |
| $\mathbb{Z}_{\leq 0}$, $\mathbb{Z}$ | ✗ |
| $[0,1]$ | ✗ $(0,1)$ has no smallest elt |

**Thm** Let $X$ be any set. There exists a well order $<$ on $X$.

(Pf uses axiom of choice)

**Def** Let $X$ be any ordered set, $x \in X$. The section is $S_x = \{y \in X : y < x\}$




**Theorem** There exists an uncountable well ordered set $S_\Omega$ such that for all $a$ in $S_\Omega$, $\{x \in S_\Omega : x < a\}$ is countable.

**Rmk** $S_\Omega$ is the "minimal uncountable well ordered set".

**Pf** Let $X$ be an uncountable well ordered set. Give $S = \{1,2\} \times X$ dictionary order. Then $S = \{1,2\} \times X$ is well ordered. Let

$$A = \{ (i,x) \in \{1,2\} \times X : S_{(i,x)} \text{ is uncountable} \}.$$

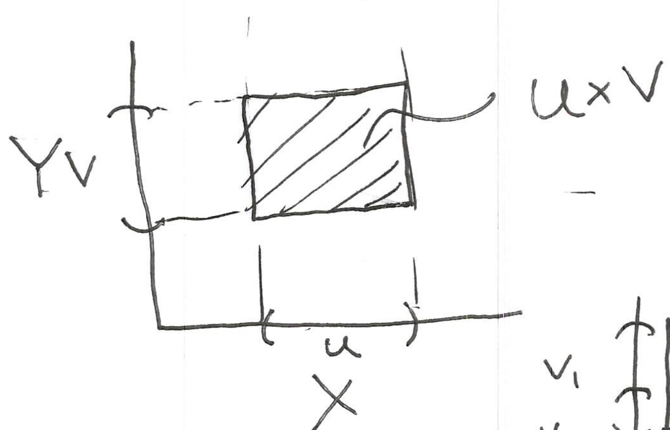
Then $A \neq \emptyset$ (b/c $(2,x) \in A$ for any $x \in X$).

Let $\Omega = \min A$. Then $S_\Omega$ satisfies the desired properties. $\square$

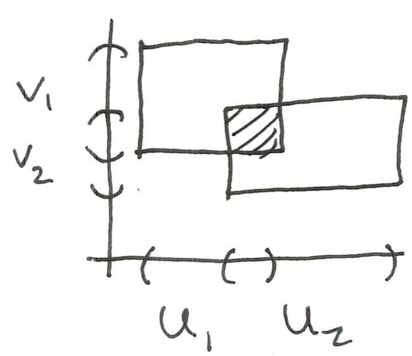
## §15 Product Topology on $X \times Y$.

**Def** Let $X, Y$ be top. spaces. The product topology on $X \times Y = \{(x,y) : x \in X, y \in Y\}$ has basis

$$\{ U \times V : U \underset{\text{open}}{\subset} X, \ V \underset{\text{open}}{\subset} Y \}$$

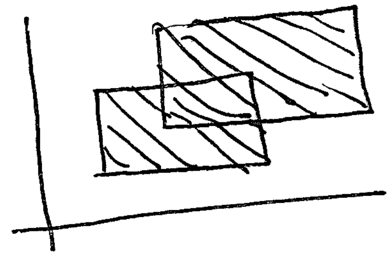



CHECK This is the basis for a topology on $X \times Y$.

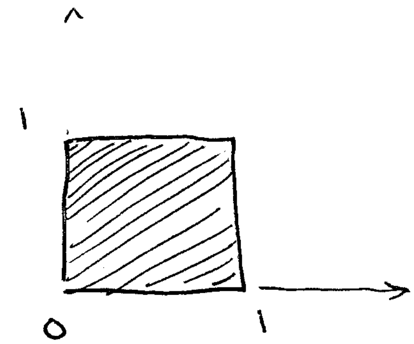



[Use
$(U_1 \times V_1) \cap (U_2 \times V_2)$
$= (U_1 \cap U_2) \times (V_1 \cap V_2)$.]

⚠ Not every open set in prod. topology on $X \times Y$ has form $U \times V$ :



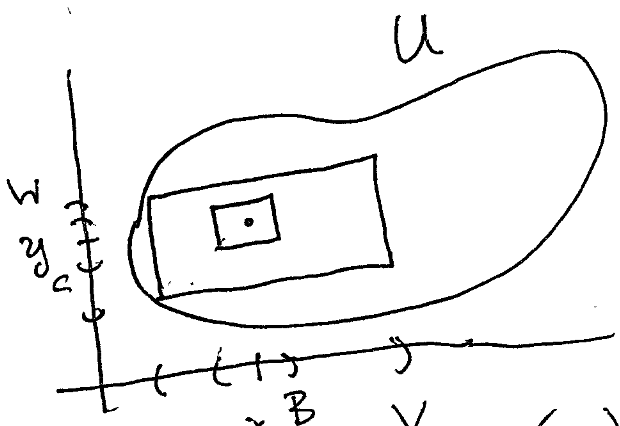
**Ex** • $I^2 = I \times I$ has product topology ~~(scribble)~~




~~Fact~~ • $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ has prod. topology = standard topology.

**Fact** If $\mathcal{B}$ is a basis for the topology of $X$,
$\mathcal{C}$ is a basis for the topology of $Y$,
then $\mathcal{D} = \{ B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C} \}$ is a basis for the topology of $X \times Y$.

**Pf** $\mathcal{D}$ is a coll'n of (basic!) open sets in $X \times Y$. Let $U \subset X \times Y$ be open & let $(x,y) \in U$. $\exists$ $V \subset X$ open st $x \in V$ & $W \subset Y$ open st $y \in W$ st $V \times W \subset U$. But $\exists$ $B \in \mathcal{B}$ st $x \in B \subset V$ & $C \in \mathcal{C}$ st $y \in C \subset W$. So $B \times C \in \mathcal{D}$ and $(x,y) \in B \times C \subset V \times W \subset U$. //



**Rmk** If $X_1, X_2, \dots, X_n$ are finitely many topological spaces, the product top. on $X_1 \times \cdots \times X_n$ has basis $\{U_1 \times \cdots \times U_n : U_i \underset{\text{open}}{\subset} X_i \text{ for } i = 1, \dots, n\}$.

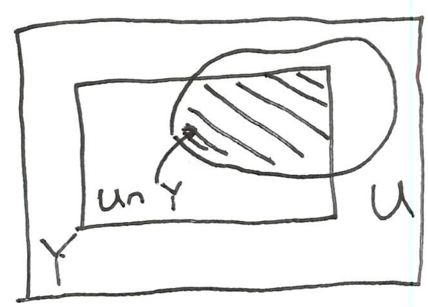
**Ex** Product topology on $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$

$= $ Euclidean top. on $\mathbb{R}^n$.

## §16 The Subspace Topology

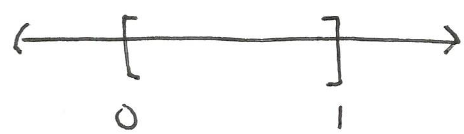
**Def** Let $X$ be a topological space and $Y \subset X$. The subspace topology on $Y$ is

$\{U \cap Y : U \subset X \text{ open}\}$.

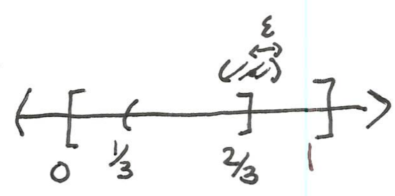



**Rmk** If $\mathcal{B}$ is a basis for top. on $X$, then $\{B \cap Y : B \in \mathcal{B}\}$ is a basis for subspace topology on $Y$.

**Ex** $X = \mathbb{R}$, $Y = [0, 1]$




| $A \subset Y$ | open in $X = \mathbb{R}$? | open in $Y = [0,1]$? |
|---|---|---|
| $(1/3, 2/3)$ | Yes. | Yes. |
| $(1/2, 1]$ | No. | Yes. $[0,1] \cap (1/2, 3/2) = (1/2, 1]$. |
| $(1/3, 2/3]$ | No. | No. |



# Important Subspaces of $\mathbb{R}^n$

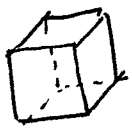
$$S^{n-1} = \{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 = 1 \}$$

$(n-1)$-dim'l sphere

$S^2 \subseteq \mathbb{R}^3$

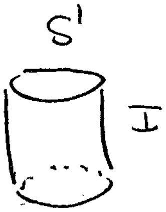
$$I^n = \{ (x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1 \}$$

$n$-dim'l cube



$I^3$

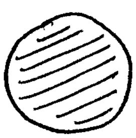
$S^1 \times I$ cylinder




$S^1 \times S^1$ torus

$$D^n = \{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 \leq 1 \}$$

~~ext~~ closed $n$-dim'l disc



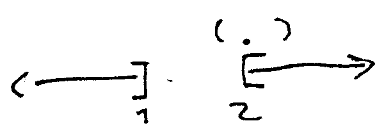
$D^2 \subseteq \mathbb{R}^2$

## §17 Closed Sets

**Def** Let $X$ be a topological space. A subset $C \subseteq X$ is closed if $X - C$ is open.

**Ex** Let ~~$X \in [0,1]$~~ $X = \mathbb{R}$

| $A$ | open? | closed? |
|---|---|---|
| $(1,2)$ | ✓ | × |
| $[1,2]$ | × | ✓ |
| $(1,2]$ | × | × |
| $\emptyset$ | ✓ | ✓ |



<u>Fact</u> Let $X$ be a top. space.

- $\emptyset, X$ are closed in $X$.
- If $\{C_\alpha\}_{\alpha \in I}$ is <u>any</u> coll-n of closed sets then $\bigcap_{\alpha \in I} C_\alpha$ is closed.
- If $C_1, \ldots, C_n$ is a <u>finite</u> list of closed sets, then $C_1 \cup \cdots \cup C_n$ is closed.

[ <u>Why?</u> $X - \bigcap_{\alpha \in I} C_\alpha = \bigcup_{\alpha \in I} (X - C_\alpha)$, $X - \left(\bigcup_{i=1}^{n} C_i\right) = \bigcap_{i=1}^{n} (X - C_i)$. ]

<u>Rmk</u> Can also define a top. on a set $X$ in terms of <u>closed</u> sets.

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Last Time

$X, Y$ top. spaces $\Rightarrow$ prod. top. on $X \times Y$ has basis $\{U \times V : \underset{\text{open}}{U \subset X}, \underset{\text{open}}{V \subset Y}\}$

$X$ top. space, $Y \subset X \Rightarrow$ Subspace topology on $Y$ is $\{U \cap Y : U \subset X \text{ open}\}$.

| $A$ | open in $X = \mathbb{R}$? | open in $Y = [0,1]$? | |
|---|---|---|---|
| $(0,1)$ | ✓ | ✓ | |
| $(0,1]$ | ✗ | ✓ | $(0,1] = Y \cap (0,2)$. |
| $[1/2, 1]$ | ✗ | ✗ | |

Fact Let $X$ be a top. space & let $Y \subset X$ be open. Give $Y$ the subspace top. Given $A \subset Y$,

$$A \text{ is open in } Y \iff A \text{ is open in } X.$$

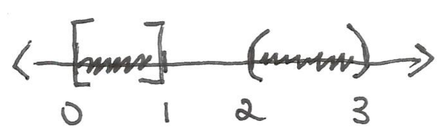
Pf $\Rightarrow$ If $A$ is open in $Y$, $\exists$ $U \subset X$ open in $X$ st $A = Y \cap U$. But $Y, U$ are open in $X$ so $A = Y \cap U$ is open in $X$.

$\Leftarrow$ If $A$ is open in $X$, $A = A \cap Y$ is open in $Y$. //

§17 Closed sets and limit points

Def Let $X$ be a topological space. A subset $C \subset X$ is closed if $X - C$ is open.

**Ex** $X = [0,1] \cup (2,3) \subset \mathbb{R}$




| $A \subset X$ | closed? in $X$ | open? in $X$ |
|---|---|---|
| $[\frac{1}{4}, \frac{3}{4}]$ | yes | no |
| $(\frac{1}{4}, \frac{3}{4})$ | no | yes |
| $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ | no | no |
| $X, [0,1], \emptyset$ | yes | yes |


**Rmk** $(2,3)$ is closed in $X$, not in $\mathbb{R}$.

"closed + open = clopen"

**Fact** Let $X$ be a top. space.

- $\emptyset, X$ are closed in $X$.
- If $\{C_\alpha\}$ is a coll'n of closed sets in $X$ then $\bigcap_\alpha C_\alpha$ is closed.
- If $\{C_1, -, C_n\}$ is a finite coll'n of closed sets in $X$ then $C_1 \cup \dots \cup C_n$ is closed.

**Why?** $X - \emptyset = X$, $X - X = \emptyset$ are open.

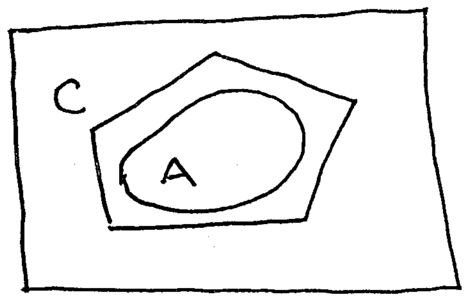
$$X - \bigcap_\alpha C_\alpha = \bigcup_\alpha (X - C_\alpha), \quad X - \bigcup_{i=1}^n C_i = \bigcap_{i=1}^n (X - C_i).$$

**Rmk** Topologies can also be defined in terms of closed sets.

**Def** Let $X$ be a topological space and let $A \subset X$.
The __closure__ $\overline{A}$ of $A$ in $X$ is

$$\overline{A} := \bigcap_{\substack{C \subset X \text{ closed} \\ A \subset C}} C .$$

nonempty $\cap$ b/c $A \subset X$ ← closed




X

**Obs** • $\overline{A}$ is closed and $A \subset \overline{A}$.

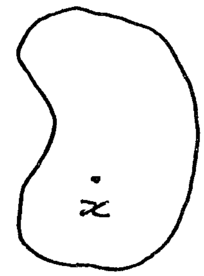
• If $A \subset C$ (closed) then $\overline{A} \subset C$.

"$\overline{A}$ is the smallest closed set containing $A$."

**Q** How to decide whether $x \in X$ lies in $\overline{A}$?

**Def** • If $X$ is a topological space and $x \in X$, a __neighborhood__ $U$ of $x$ is an open set $U \subset X$ with $x \in U$.

U

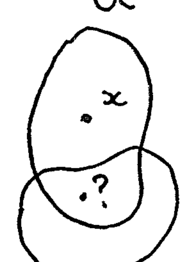



**Fact** Let $X$ be a topological space and let $A \subset X$.
Let $x \in X$.

① $x \in \overline{A} \iff$ for every nbhd $U$ of $x$, $\underbrace{U \cap A \neq \emptyset}_{\text{"}U \text{ meets } A\text{"}}$

② If the topology on $X$ is given by a basis $\mathcal{B}$,
$x \in \overline{A} \iff$ every basis elt $B \in \mathcal{B}$ st $x \in B$ has $A \cap B \neq \emptyset$.

U




A

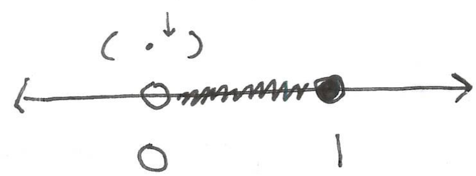
Pf of ①

$\Rightarrow$ Let $x \in \overline{A}$ and let $U$ be a nbhd of $x$.
Then $X - U$ is closed in $X$ and $x \notin X - U$.
B/c $x \in \overline{A} = \bigcap_{\substack{A \subset C \\ C \text{ closed}}} C$, we must have $A \not\subset X - U$.
But then $A \cap U \neq \emptyset$.

$\Leftarrow$ Suppose $x \notin \overline{A} = \bigcap_{\substack{A \subset C \\ C \text{ closed}}} C$. There exists $C \subset X$ closed with $A \subset C$ such that $x \notin C$. Then $U = X - C$ is a nbhd of $x$ s.t. $U \cap A = \emptyset$. ∎

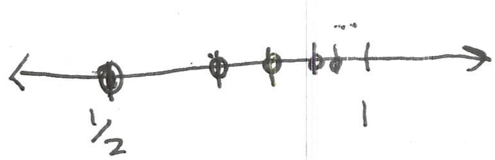
Ex • Let $X = \mathbb{R}$, std topology.

$\overline{(0,1]} = [0,1]$.



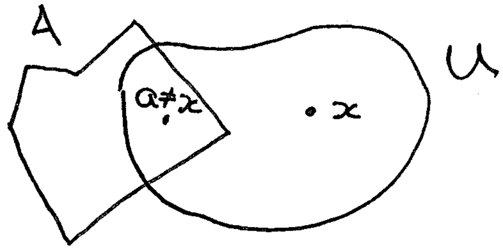

$\overline{[0,1]} = [0,1]$ ($\overline{C} = C$ for any closed set $C$.)

$\overline{\{1/2, 3/4, 7/8, 15/16, \dots\}} = \{1\} \cup \{1/2, 3/4, 7/8, \dots\}$.




$\overline{\mathbb{Q}} = \mathbb{R}$. (for all $x \in \mathbb{R}$, for all $(a,b)$ st $x \in (a,b)$, $\exists\, q \in \mathbb{Q}$ st $q \in (a,b)$.)

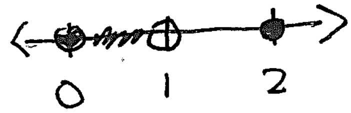
**Def** Let $X$ be a topological space and let $A \subset X$. A *limit point* of $A$ is a pt $x \in X$ st every nbhd $U$ of $x$ contains a point $a \in A$ $\boxed{\text{with } a \neq x.}$




$A' := \{\text{all limit points of } A\}$.

**Ex** $X = \mathbb{R}$, standard topology.

$A = [0,1) \cup \{2\}$




| $x$ | limit pt of $A$? |
|---|---|
| 0 | yes |
| 1 | yes |
| -1 | no eg $(-3/2, -1/2)$ |
| 2 | no eg $(3/2, 5/2)$. |

~~Theorem~~ Fact Let $X$ be a topological space & let $A \subset X$. We have $\overline{A} = A \cup A'$.

Pf "$\subset$" Let $x \in \overline{A} - A$. Then if $U$ is a nbhd of $x$, $U \cap A \neq \emptyset$. Since $x \notin A$, $U \cap (A - \{x\}) \neq \emptyset$ so $x \in A'$.

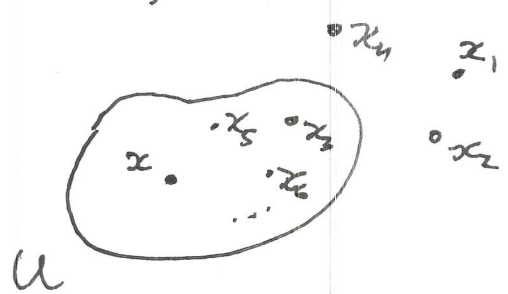
"$\supset$" Clearly $A \subset \overline{A}$. If $x \in A'$, every nbhd of $x$ meets $A$, so $x \in \overline{A}$. //

~~**Def** Let $X$ be a top. space. $X$ is Hausdorff if for all $x, y \in X$ with $x \neq y$ $\exists$ nbhds $U$ of $x$ and $V$ of $y$ with $U \cap V$~~

**Def** A sequence in a top space $X$ is a function

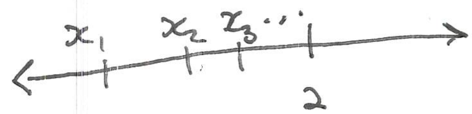
$$\mathbb{Z}_{>0} \longrightarrow X$$
$$n \longmapsto x_n.$$

Given $x \in X$, we say $x_n$ converges to $x$ (and write $x_n \longrightarrow x$) if for any nbhd $U$ of $x$, $\exists$ $N > 0$ s.t. $n > N \Rightarrow x_n \in U$.




**Ex** $X = \mathbb{R}_\ell$

$x_n = 2 - 1/n$




CLAIM $x_n$ does not converge to any pt in $\mathbb{R}_\ell$.

Let $a \in \mathbb{R}_\ell$. If $a < 2$ then $\exists$ $N > 0$ st $2 - \frac{1}{N} > a$. Then $[a, 2 - \frac{1}{N})$ is a nbhd of $a$ & $x_n \notin [a, 2 - \frac{1}{N})$ $\forall$ $n > N$, so $x_n \not\rightarrow a$. If $a \geq 2$, $[a, a+1)$ is a nbhd of $a$ & $x_n \notin [a, a+1)$ for all $n \geq 1$.

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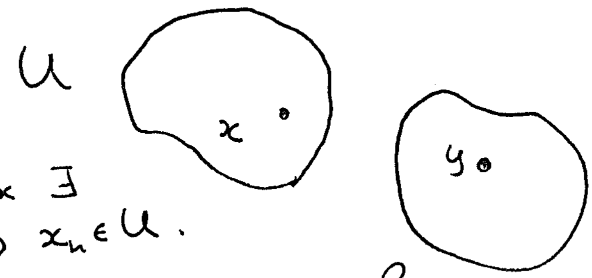
**Last Time** - $C \subset X$ is closed $\iff$ $X - C$ is open

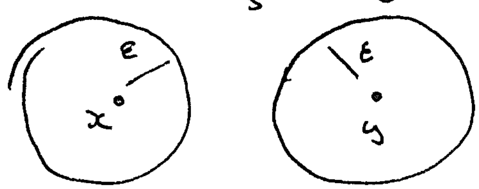
- $\overline{A} = \bigcap_{\substack{A \subset C \\ C \text{ closed}}} C$.




- $x_n \to x \iff$ for every nbhd $U$ of $x$ $\exists$ $N > 0$ st $n \geq N \Rightarrow x_n \in U$.

- $X$ is Hausdorff $\iff$ for all $x \neq y$ in $X$, $\exists$ nbhds $U$ of $x$ & $V$ of $y$ st $U \cap V = \emptyset$ V

**Ex** - $\mathbb{R}^n$ is Hausdorff: If $x \neq y$ in $\mathbb{R}^n$, let $\varepsilon := \frac{1}{3} d(x,y)$. Then $B_d(x,\varepsilon) \cap B_d(y,\varepsilon) = \emptyset$.




- $\mathbb{R}_\ell$ is Hausdorff (finer than std. top.)
- $\mathbb{R}$ in finite comp. top. is NOT Hausdorff.
  Eg if $U$ is a nbhd of 0, $V$ a nbhd of 1, then $(\mathbb{R}-U) \cup (\mathbb{R}-V) = \mathbb{R} - (U \cap V)$ is finite so $U \cap V \neq \emptyset$.

**Fact** Let $X$ be a Hausdorff space & let $x_n$ be a sequence in $X$. Then $x_n$ converges to at most one element of $X$.

**Pf** Let $x, y \in X$ & suppose $x_n \to x$ & $x_n \to y$. If $x \neq y$ choose nbhds $U, V$ of $x, y$ st $U \cap V = \emptyset$. $\exists$ $N$ st $n \geq N \Rightarrow x_n \in U$ and $x_n \in V$. So $x_N \in U \cap V = \emptyset$. $\divideontimes$

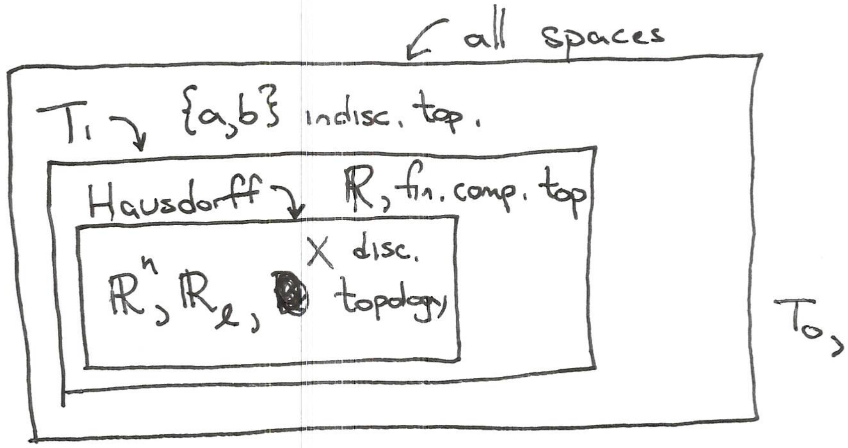
**Fact** If $X$ is a Hausdorff space & $x \in X$, then $\{x\}$ is closed.

**Pf** Let $y \neq x$. $\exists$ nbhds $U_{xy}$ of $x$ & $V_y$ of $y$ st $U_{xy} \cap V_y = \emptyset$. Then $y \in V_y \subset X - \{x\}$ so
$$X - \{x\} = \bigcup_{y \neq x} V_y$$
is open & $\{x\}$ is closed. //

**Rmk** Converse is not true.
$\mathbb{R}$, fin. comp top has 1-pt sets closed, but not Hausd.

**Def** A topological space $X$ is $\underline{T_1}$ if for all $x \in X$, $\{x\}$ is closed.

**Ex**



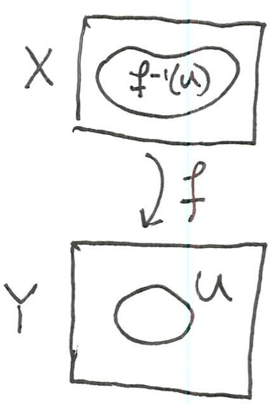

Hausdorff $\equiv T_2$

| Kolmogorov | Fréchet | Hausdorff | Urysohn | regular Hausd. | Tychonoff | normal Hausd. | completely norm. Hausd. | perfectly norm Hausd. |
|---|---|---|---|---|---|---|---|---|

$$T_0 \supset T_1 \supset T_2 \supset T_{2\frac{1}{2}} \supset T_3 \supset T_{3\frac{1}{2}} \supset T_4 \supset T_5 \supset T_6$$

## §18 Continuous Functions

**Def** Let $X, Y$ be topological spaces. A function $f: X \to Y$ is continuous if for all $U \subset Y$ open, $f^{-1}(U) \subset X$ is open.



Ex

- $\mathrm{id}_X : X \longrightarrow X$, $x \longmapsto x$ is cts

- $f : \mathbb{R} \longrightarrow \mathbb{R}_\ell$, $x \longmapsto x$ is NOT cts: $f^{-1}([0,1)) = [0,1)$.

  open in $\mathbb{R}_\ell$ — NOT open in $\mathbb{R}$

- $g : \mathbb{R}_\ell \longrightarrow \mathbb{R}$, $x \longmapsto x$ is cts "$\mathbb{R}_\ell$ is finer than $\mathbb{R}$, std top."

- If $f : X \longrightarrow Y$ & $g : Y \longrightarrow Z$ are cts, so is $g \circ f : X \longrightarrow Z$: ( $(g\circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ for $U \subset Z$ open).

$\Rightarrow$ We have a category of
- Topological Spaces $X$
- Continuous maps $X \xrightarrow{f} Y$.

Rmk If $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ (std topologies) then

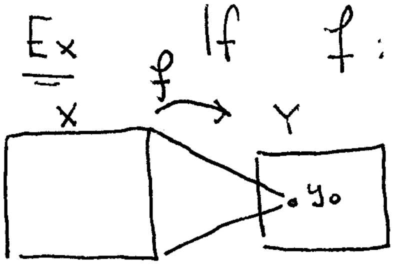
$f : X \longrightarrow Y$ is CTS $\iff$ $\forall\ \varepsilon > 0$, $\forall\ x \in X$, $\exists\ \delta > 0$ st $\forall\ x' \in X$ w/ $d(x,x') < \delta$ we have $d(f(x), f(x')) < \varepsilon$.

eg $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$(x,y) \longmapsto (1, x^2 - 2xy, \cos(x^3) + \sin(y))$

is continuous. "b/c calculus"

Ex If $f : X \to Y$, $x \mapsto y_0$ for some $y_0 \in Y$ then $f$ is CTS



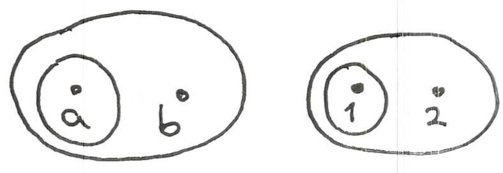

If $U \subset Y$ is open, $f^{-1}(U) = \begin{cases} X & y_0 \in U \\ \emptyset & y_0 \notin U, \end{cases}$

so $f^{-1}(U)$ is open in $X$.

**Q** When are two spaces $X, Y$ "the same"?

**Def** A continuous fcn $f: X \longrightarrow Y$ is a homeomorphism if $f$ is bijective and $f^{-1}: Y \longrightarrow X$ is also continuous.
Two spaces $X, Y$ are homeomorphic if $\exists$ a homeomorphism $f: X \longrightarrow Y$.

**Ex**

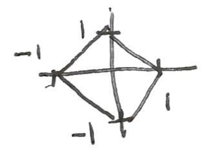


$X$ $\qquad$ $Y$

$$f: X \longrightarrow Y$$
$$a \longmapsto 1$$
$$b \longmapsto 2$$
is a homeom.

**CLAIM** If $X = S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

$Y = \{(x,y) \in \mathbb{R}^2 : |x| + |y| = 1\}$



then $X$ and $Y$ are homeomorphic.

**Pf** Define $f: X \longrightarrow Y$ by $f(x,y) = \left(\dfrac{x}{|x|+|y|}, \dfrac{y}{|x|+|y|}\right)$.
Then $f$ is CTS (by calculus) & well-defined
$\left(\text{b/c } |x|+|y| \neq 0 \ \forall (x,y) \in X = S^1 \ \& \ \left|\dfrac{x}{|x|+|y|}\right| + \left|\dfrac{y}{|x|+|y|}\right| = 1\right)$.
Define $g: Y \longrightarrow X$ by $g(x,y) = \left(\dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}}\right)$.
Then $g$ is CTS (by calculus) & well-defined
$\left(\text{b/c } \forall (x,y) \in Y,\ x^2+y^2 \neq 0 \ \& \ \left(\dfrac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\dfrac{y}{\sqrt{x^2+y^2}}\right)^2 = 1\right)$.
If $(x,y) \in X$ then $gf(x,y) = g\left(\dfrac{x}{|x|+|y|}, \dfrac{y}{|x|+|y|}\right) = \left(\dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}}\right) = (x,y)$.
If $(x,y) \in Y$ then $fg(x,y) = f\left(\dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}}\right) = \left(\dfrac{x}{|x|+|y|}, \dfrac{y}{|x|+|y|}\right) = (x,y)$.
So $f: X \to Y$ is a homeomorphism. //

⚠ Define $f : [0,1) \longrightarrow S^1$ by

$$x \longmapsto e^{2\pi i x}$$

$f$ is a CTS bijn but <u>NOT</u> a <u>homeomorphism</u>!

($f^{-1}$ is not continuous  .)


<u>Q</u> Are $[0,1)$ and $S^1$ homeomorphic? No, but <u>WHY</u>?

<u>Ex</u> • If $Y \subset X$ is a subspace, the inclusion

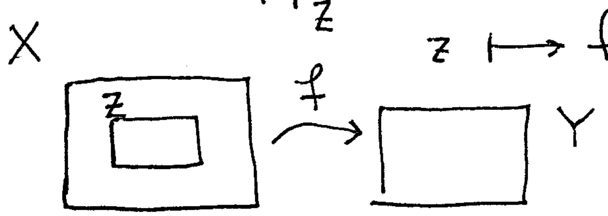
$$\iota : Y \to X, \quad y \mapsto y \quad \text{is CTS.}$$

<u>Why</u>? For $U \subset X$ open, $\iota^{-1}(U) = \overbrace{U \cap Y}^{\text{open in subsp top on } Y}$.

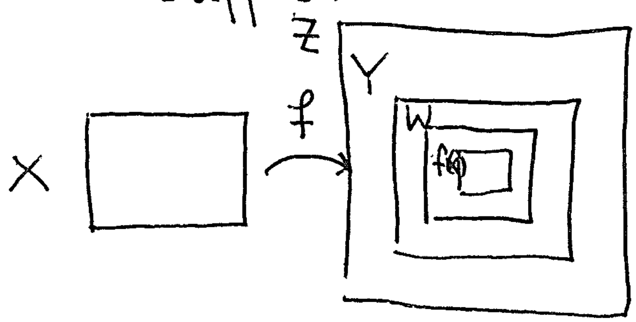
• If $f : X \to Y$ is CTS & $Z \subset X$, then

$$f|_Z : Z \to Y, \quad z \mapsto f(z) \quad \text{is CTS.}$$

(If $\iota : Z \hookrightarrow X$ is incl, $f|_Z = f \circ \iota$.)




• Suppose $f : X \to Y$ is CTS & $f(X) \subset W \subset Y \subset Z$.




Then
$$\begin{cases} X \to W, & x \mapsto f(x) \\ X \to Z, & x \mapsto f(x) \end{cases}$$

are continuous.