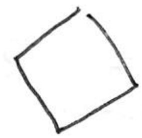
Math 190 Lecture 7 1/22/2014

$X = S^1 = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$ 

$Y = \{ (x,y) \in \mathbb{R}^2 : |x| + |y| = 1 \}$ 

CLAIM X is homeo. to Y.

Pf Define $f : X \to Y$, $(x,y) \mapsto \frac{1}{|x|+|y|}(x,y)$, $g : Y \to X$, $(x,y) \mapsto \frac{1}{\sqrt{x^2+y^2}}(x,y)$.

f, g are well-defined & CTS (b/c calculus).

If $(x,y) \in X$ then

$$gf(x,y) = g\left(\frac{1}{|x|+|y|}(x,y) \right) \overset{x^2+y^2=1}{=} (x,y).$$

$$fg(x,y) = f\left(\frac{1}{\sqrt{x^2+y^2}}(x,y) \right) \underset{|x|+|y|=1}{=} (x,y).$$

So $f = g^{-1}$ & f is a homeo. //

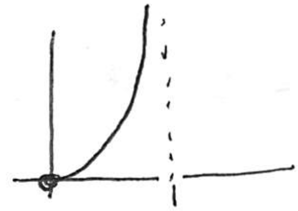
Fact If X, Y are top spaces, $y_0 \in Y$, & $f : X \to Y$, $x \mapsto y_0$, f is CTS.

$$f^{-1}(U) = \begin{cases} X & y_0 \in U \\ \emptyset & y_0 \notin U. \end{cases}$$

Fact If $A \subset X$ is a subsp., $\iota : A \hookrightarrow X$, $x \mapsto x$ is CTS.

$\ulcorner$ $\iota^{-1}(U) = U \cap A$ $\lrcorner$

Fact $[0,1) \underset{\text{homeo.}}{\cong} \mathbb{R}_{\geq 0}$.



$\ulcorner$**WHY?**

$$f : [0,1) \to \mathbb{R}_{\geq 0}, \quad \theta \mapsto \tan\left(\tfrac{\pi}{2}\theta\right) \qquad g : \mathbb{R}_{\geq 0} \to [0,1), \quad x \mapsto \tfrac{2}{\pi}\arctan(x)$$

f, g are CTS (b/c calculus) & mutually inverse

So f is a homeom. $\lrcorner$

Fact. Suppose $f : X \to Y$ is CTS & $Z \subseteq X$.

Then $f|_Z : Z \to Y$, $x \mapsto f(x)$ is CTS.

FACT Suppose $f : X \to Y$ is CTS & $f(X) \subset W \subset Y \hookrightarrow Z$.

Then $X \to W$, $x \mapsto f(x)$, $X \to Z$, $x \mapsto f(x)$ are CTS.

Rmk Let $f : X \to Y$ be a map of top. spaces.

f is CTS $\iff$ for any **closed** set $C \subset Y$, $f^{-1}(C)$ is closed in X.

⌜ Why? $X - f^{-1}(C) = f^{-1}(Y - C)$. ⌟

Ex $\mathrm{Mat}_{n\times n}(\mathbb{R}) = \{\text{all } n\times n \text{ matrices w/ entries in } \mathbb{R}\} = \mathbb{R}^{n^2}$

$$\begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}.$$

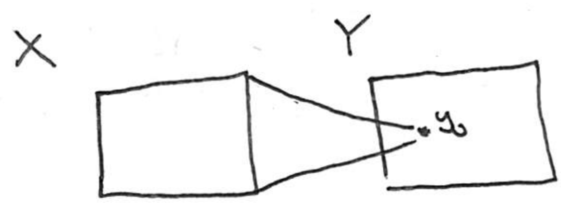
The <u>determinant</u> $\det : \mathrm{Mat}_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$ is CTS.

$$A \longmapsto \det A$$

⌜ <u>Why?</u> $\det A = \sum_{\sigma\in S_n} (\operatorname{sign}\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$. ⌟ a <u>polynomial</u> in entries of A!

So $GL_n(\mathbb{R}) = \{A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) : \det A \neq 0\}$

$$= (\det)^{-1}\underbrace{(\mathbb{R} - \{0\})}_{\text{open in } \mathbb{R}}$$



is open in $\mathrm{Mat}_{n\times n}(\mathbb{R})$!

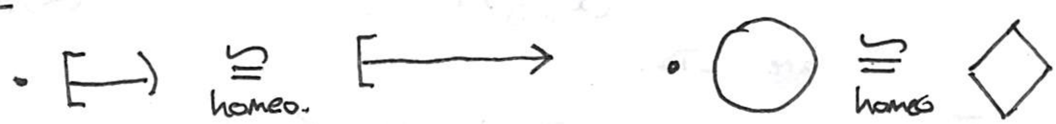
FACT ~~Let~~ X, Y be top. spaces, $y_0 \in Y$.

Then $f : X \to Y$, $x \mapsto y_0$ is CTS.

⌜ $f^{-1}(U) = \begin{cases} X & y_0 \in U \\ \varnothing & y_0 \notin U \end{cases}$ both open in X! ⌟

Math 190 Lecture 8 1/28/2018

Last Time • $GL_n(\mathbb{R}) \subseteq Mat_n(\mathbb{R})$ is open.



More on CTS functions

Pasting Lemma Suppose $X = A \cup B$ where A, B are closed sets. Let $f: A \to Y$, $g: B \to Y$ be CTS fcns s.t. $f(x) = g(x)$ for all $x \in A \cap B$. The fcn $h: X \to Y$ given by $h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$

is CTS.

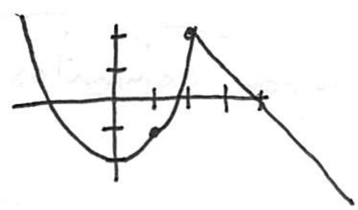
⌜ If $C \subseteq Y$ is closed, $h^{-1}(C) = [f^{-1}(C) \cup g^{-1}(C)]$

$f^{-1}(C)$: closed in A, hence X b/c A closed

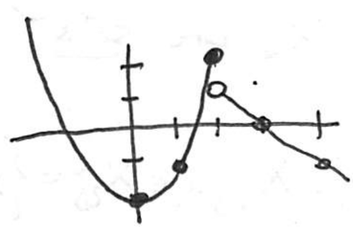
$g^{-1}(C)$: closed in B, hence X b/c B closed. ⌟

Ex Consider

$$f(x): \mathbb{R} \to \mathbb{R} \quad f(x) = \begin{cases} x^2 - 2 & x \leq 2 \\ -x+4 & x \geq 2 \end{cases}$$



Then f is CTS.



$$h(x) = \begin{cases} x^2 - 2 & x \leq 2 \\ -x+3 & x > 2 \end{cases}$$

not closed!

is NOT cts

Fact If $X \times Y$ is a product, the projns

$$\pi_X : X \times Y \twoheadrightarrow X, \quad \pi_Y : X \times Y \twoheadrightarrow Y$$
$$(x,y) \longmapsto x \qquad\qquad (x,y) \longmapsto y$$

are CTS.

$\ulcorner$ If $U \subset X$, $\pi_X^{-1}(U) = U \times Y$. $\lrcorner$

Fact Suppose $f: A \to X$, $g: A \longrightarrow Y$ are CTS. Then

$$h: A \longrightarrow X \times Y$$
$$a \longmapsto \cancel{(x,y)}\ (f(a), g(a))$$

is CTS.

Pf Let $U \times V$ be a basis open set in $X \times Y$, so $U \subset X$ is open & $V \subset Y$ is open. Then $h^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V)$ is open in A. $\blacksquare$

"Universal Property of $X \times Y$."

$$\begin{tikzcd}
A \arrow[r, dotted, "\exists! h"] \arrow[rr, bend left, "g"] \arrow[dr, bend right, "f"'] & X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \\
& X &
\end{tikzcd}$$

If $f: A \to X$ & $g: A \to Y$ are CTS, $\exists!$ CTS fcn $h: A \to X \times Y$ st

$$\underbrace{g = \pi_Y \circ h, \quad f = \pi_X \circ h}_{\text{"the diagram commutes."}}$$

§ 19 ~~18~~. Arbitrary products

Q What is $\prod_{\alpha \in J} X_\alpha$ if X_α are sets?

Well, $X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\} = \{x : \{1,2\} \to X_1 \cup X_2 : x(i) \in X_i\}$.

Rmk If J is finite, product & box topologies on $\prod_{\alpha \in J} X_\alpha$ are the same.

FACT Fix $\alpha_0 \in J$ & define $\pi_{\alpha_0} : \prod_{\alpha \in J} X_\alpha \twoheadrightarrow X_{\alpha_0}$

$$(x_\alpha)_{\alpha \in J} \longmapsto x_{\alpha_0}.$$

Then π_{α_0} is CTS if $\prod_{\alpha \in J} X_\alpha$ is given the product or box topologies.

Why? If $U_{\alpha_0} \subset X_{\alpha_0}$ then $\pi_{\alpha_0}^{-1}(U_{\alpha_0}) = \prod_{\alpha \in J} V_\alpha$

where $V_\alpha = \begin{cases} X_\alpha & \alpha \neq \alpha_0 \\ U_{\alpha_0} & \alpha = \alpha_0 . \end{cases}$

Ex ~~Therefore~~ $\mathbb{R}^\omega = \mathbb{R}^{\mathbb{Z}_{>0}} = \{\text{all real sequences } (a_1, a_2, \dots)\}$.

Define $f : \mathbb{R} \longrightarrow \mathbb{R}^\omega$ by

$$t \longmapsto (t, t, t, \dots)$$

CLAIM f is NOT CTS if $\mathbb{R}^\omega$ has the box topology.

Pf The set $\underbrace{(-1,1) \times (-\tfrac{1}{2}, \tfrac{1}{2}) \times (-\tfrac{1}{3}, \tfrac{1}{3}) \times \dots}_{U} \subset \mathbb{R}^\omega$ is open. We have

$$f^{-1}(U) = (-1,1) \cap (-\tfrac{1}{2}, \tfrac{1}{2}) \cap (-\tfrac{1}{3}, \tfrac{1}{3}) \cap \dots$$

$= \{0\}$, which is NOT open in $\mathbb{R}$.

Similarly,

$$X_1 \times \cdots \times X_n = \{\text{all fcns } x : \{1, \ldots, n\} \to X_1 \cup \cdots \cup X_n : \overset{\substack{x_i \\ \shortparallel}}{x(i)} \in X_i\}.$$

If J is any "index" set & X_α is a set $\forall\, \alpha \in J$,

then

$$\prod_{\alpha \in J} X_\alpha = \Big\{\text{all fcns } x : J \to \bigcup_{\alpha\in J} X_\alpha : \begin{array}{c}\text{for all } \alpha \in J, \\ \underset{\substack{\shortparallel \\ x_\alpha}}{x(\alpha)} \in X_\alpha\end{array}\Big\}.$$

"tuple" notation $(x_\alpha)_{\alpha \in J}$.

Important Special Case $X_\alpha = X$ for all α:

$$\prod_{\alpha \in J} X \equiv X^J = \{\text{all fcns } J \to X\}$$

eg $\mathbb{R}^{\mathbb{R}} = \{\text{all fcns } f : \mathbb{R} \to \mathbb{R}\}$

$\mathbb{R}^\omega \overset{\text{def}}{=} \mathbb{R}^{\mathbb{Z}_{>0}} = \{\text{all sequences } (x_1, x_2, \ldots) \text{ in } \mathbb{R}\}$

$\{0,1\}^{\mathbb{Z}_{>0}} = \{\text{all infinite } \overset{0,1}{\text{binary}} \text{ sequences } (b_1, b_2, \ldots)\}.$

Q If the X_α are spaces, how to put a topology on $\prod_{\alpha \in J} X_\alpha$?

Bad Answer The box topology on $\prod_{\alpha \in J} X_\alpha$ has basis

$$\Big\{\prod_{\alpha \in J} U_\alpha : U_\alpha \subset X_\alpha \text{ open for all } \alpha \in J\Big\}.$$

Good Answer The product topology on $\prod_{\alpha \in J} X_\alpha$ has

basis $\Big\{\prod_{\alpha \in J} U_\alpha : \begin{array}{l} U_\alpha \subset X_\alpha \text{ open } \forall\, \alpha \in J \;\& \\ U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \end{array}\Big\}.$

Math 190 1/29/2018 Lecture 9

Last Time

If X_α is a set for all $\alpha \in I$, $\prod_{\alpha\in I} X_\alpha = \left\{ \text{all fcns } x : I \to \bigcup_\alpha X_\alpha : \underset{x(\alpha)}{x_\alpha} \in X_\alpha \ \forall \alpha \in I \right\}$

If X_α is a space for all $\alpha \in I$,

- box topology on $\prod_{\alpha \in I} X_\alpha$ has basis $\left\{ \prod_{\alpha\in I} U_\alpha : \begin{array}{l} U_\alpha \subset X_\alpha \text{ open} \\ \text{for all } \alpha \in I \end{array} \right\}$
- product topology on $\prod_{\alpha\in I} X_\alpha$ has basis $\left\{ \prod_{\alpha \in I} U_\alpha : \begin{array}{l} U_\alpha \subset X_\alpha \text{ open } \forall \alpha \in I, \\ U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \end{array} \right\}$.

~~Rmk If $\mathcal{B}$ is a basis for~~

Fact For any $\alpha_0 \in I$, let $\prod_{\alpha\in I} X_\alpha \xrightarrow{\pi_{\alpha_0}} X_{\alpha_0}$, $(x_\alpha)_{\alpha\in I} \longmapsto x_{\alpha_0}$ be the projn. Then π_{α_0} is CTS ~~when~~ if $\prod_\alpha X_\alpha$ is given the product OR box topologies.

$\ulcorner$ If $U_{\alpha_0} \subset X_{\alpha_0}$ is open, $\pi_{\alpha_0}^{-1}(U_{\alpha_0}) = \prod_{\alpha\in I} V_\alpha$ where

$$V_\alpha = \begin{cases} X_\alpha & \alpha \neq \alpha_0 \\ U_{\alpha_0} & \alpha = \alpha_0, \end{cases}$$

so $\prod_\alpha V_\alpha$ is a basic open set in either prod. or box top on $\prod_\alpha X_\alpha$. $\lrcorner$

Recall If $\mathbb{R}^\omega = \{ (a_1, a_2, \dots) : a_n \in \mathbb{R} \} \ (= \mathbb{R}^{\mathbb{Z}_{>0}})$ is given box topology,

$$f : \mathbb{R} \longrightarrow \mathbb{R}^\omega, \quad t \longmapsto (t, t, \dots)$$

is NOT cts. $\ulcorner$ $f^{-1}\left((-1,1) \times (-\tfrac{1}{2}, \tfrac{1}{2}) \times (-\tfrac{1}{3}, \tfrac{1}{3}) \times \cdots \right)$

$= \bigcap_{n>0} (-\tfrac{1}{n}, \tfrac{1}{n}) = \{0\}$. $\lrcorner$

Theorem

Suppose X_α is a top space $\forall\ \alpha \in I$ & give $\prod_{\alpha\in I} X_\alpha$ the product topology. If $f_\alpha : A \to X_\alpha$ is a CTS fcn $\forall\ \alpha \in I$, then

$$f : A \longrightarrow \prod_{\alpha \in I} X_\alpha$$
$$a \longmapsto (f_\alpha(a))_{\alpha\in I} \quad \text{is CTS.}$$

Pf Let $\prod_{\alpha\in I} U_\alpha$ be a basic open set in $\prod_{\alpha \in I} X_\alpha$, so $U_\alpha \subset X_\alpha$ is open $\forall\ \alpha$.

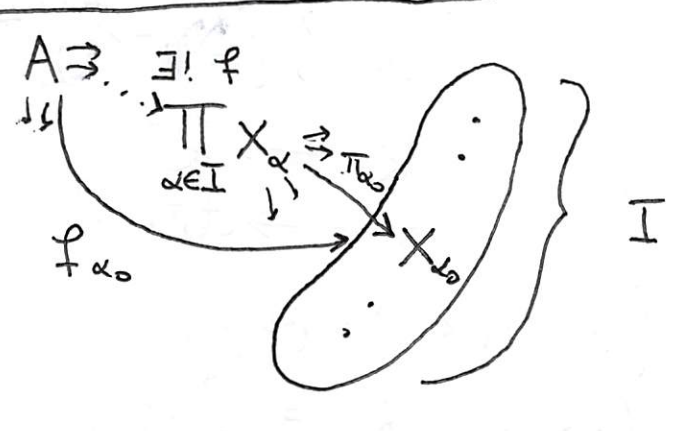
Then $\exists\ J \subset I$ FINITE st $U_\alpha = X_\alpha$ unless $\alpha \in J$. We have

$$f^{-1}\Big(\prod_{\alpha\in I} U_\alpha\Big) = \bigcap_{\alpha\in I} f_\alpha^{-1}(U_\alpha) \overset{U_\alpha = X_\alpha \text{ if } \alpha\notin J}{=} \bigcap_{\alpha \in J} \overbrace{f_\alpha^{-1}(U_\alpha)}^{\text{open, b/c } f_\alpha \text{ CTS}},$$

which is open in A b/c J is finite. $\blacksquare$

⌜ So $\mathbb{R} \longrightarrow \mathbb{R}^\omega$, $t \longmapsto (t, t, \ldots)$ is CTS if $\mathbb{R}^\omega$ has prod top. ⌟

Universal Property of P.T



<u>Fact</u> If X_α is Hausdorff for all $\alpha \in I$, so is $\prod_{\alpha \in I} X_\alpha$ in box or product topologies.

⌜ If $(x_\alpha)_{\alpha \in I} \neq (y_\alpha)_{\alpha \in I}$ in $\prod_\alpha X_\alpha$, $\exists\ \alpha_0$ st $x_{\alpha_0} \neq y_{\alpha_0}$. B/c X_{α_0} is <u>Hausdorff</u>, $\exists\ U, V \subset X_{\alpha_0}$ open st $x_{\alpha_0} \in U$, $y_{\alpha_0} \in V$, and $U \cap V = \varnothing$.

Now $\prod_\alpha U_\alpha$ is a nbhd of $(x_\alpha)_{\alpha \in I}$

$\prod_\alpha V_\alpha$ —— " —— $(y_\alpha)_{\alpha \in I}$, where

$$U_\alpha = \begin{cases} X_\alpha & \alpha \neq \alpha_0 \\ U & \alpha = \alpha_0 \end{cases}, \qquad V_\alpha = \begin{cases} X_\alpha & \alpha \neq \alpha_0 \\ V & \alpha = \alpha_0 \end{cases} \quad \&$$

$$\left(\prod_\alpha U_\alpha\right) \cap \left(\prod_\alpha V_\alpha\right) = \varnothing. \quad \blacksquare$$ ⌟

<u>Ex</u> Give $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots$ the <u>Box</u> topology.

Consider the sequence

$$\vec{x}_1 = (1, 1, 1, 1, \ldots)$$
$$\vec{x}_2 = (1/2, 1/2, 1/2, 1/2, \ldots)$$
$$\vec{x}_3 = (1/3, 1/3, 1/3, 1/3, \ldots) \quad \ldots$$

<u>CLAIM</u> $\vec{x}_n \not\to \vec{0} \equiv (0,0,0,\ldots)$ in $\mathbb{R}^\omega$.

<u>Pf</u> Consider the nbhd $U = (-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots$ of $\vec{0}$ in $\mathbb{R}^\omega$. We have $\vec{x}_n \notin U$ for all $n \geq 1$, so $\vec{x}_n \not\to \vec{0}$. //

⌜ $\vec{x}_n$ does not converge in $\mathbb{R}^\omega$ in box top. ⌟

Fact If $A_\alpha \subset X_\alpha$ for all $\alpha \in I$,

$$\overline{\prod_{\alpha \in I} A_\alpha} = \prod_{\alpha \in I} \overline{A_\alpha} \quad \text{in product or box topologies.}$$

Why? Let $(x_\alpha)_{\alpha \in I}$ be a point in $\prod_\alpha X_\alpha$.

Then $(x_\alpha) \in \overline{\prod_{\alpha \in I} A_\alpha} \iff$ every nbhd U of (x_α) has $\prod_{\alpha \in I} A_\alpha \cap U \neq \emptyset$.

$\iff$ every basic open nbhd $\prod_\alpha U_\alpha$ of (x_α) has $\left(\prod_\alpha A_\alpha\right) \cap \left(\prod_\alpha U_\alpha\right) \neq \emptyset$

$\iff$ for every $\alpha \in I$, every nbhd U_α of x_α in X_α has $U_\alpha \cap A_\alpha \neq \emptyset$

$\iff$ for every $\alpha \in I$, $x_\alpha \in \overline{A_\alpha}$

$\iff$ $(x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} \overline{A_\alpha}$. ∎

§ 20 The Metric Topology

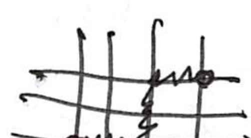
Def Let X be a set. A <u>metric</u> on X is a fcn $d: X \times X \longrightarrow \mathbb{R}_{\geq 0}$ s.t.

- $d(x,y) = 0 \iff x = y$
- $d(x,y) = d(y,x)$ for all $x, y \in X$,
- $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$. "Triangle Inequality"

Eg $X = \mathbb{R}^n$

Standard metric: $d(x,y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}$

Manhattan metric $d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$



* If $d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ is a metric, $x \in X$, $\varepsilon > 0$, the <u>open</u> <u>ball</u> is

$$B_d(x,\varepsilon) := \{ y \in X : d(x,y) < \varepsilon \}.$$

<u>Def</u> Let d be a metric on a set X. The <u>metric</u> <u>topology</u> on (X,d) has basis $\{ B_d(x,\varepsilon) : x \in X, \varepsilon > 0 \}$.

<u>Ex</u> ① $X \equiv$ any set. The <u>discrete</u> <u>metric</u> $d : X \times X \to \mathbb{R}_{\geq 0}$

st $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$

So $B_d(x,1) = \{x\}$ for all $x \in X$ &

metric top. on (X,d) $=$ discrete top. on X.

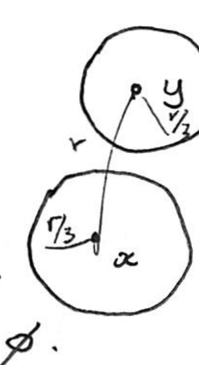
② $X = \mathbb{R}^n$

metric top on $(\mathbb{R}^n, d_{standard})$ $=$ Euclidean top on $\mathbb{R}^n$ $=$ metric top. on $(\mathbb{R}^n, d_{Manhattan})$.

<u>FACT</u> Let (X,d) be a <u>metric</u> <u>space</u>.

(ie d is a Metric on X, X has metric topology.)

Then X is <u>Hausdorff</u>.



<u>Pf</u> Let $x \neq y$ in X & let $r = d(x,y) > 0$.

If $\varepsilon = r/3$, we claim $B(x,\varepsilon) \cap B(y,\varepsilon) = \emptyset$.

If $z \in B(x,\varepsilon) \cap B(y,\varepsilon)$, then

$$r = d(x,y) \leq d(x,z) + d(z,y) < \varepsilon + \varepsilon = 2\varepsilon = \frac{2r}{3}. \quad \divideontimes$$

$\Rightarrow$ If $|X| \geq 2$, indisc. topology on X can <u>never</u> be realized with a metric.

<u>Def</u> Let X be a topological space. X is <u>metrizable</u> if $\exists$ a metric d on the <u>set</u> X s.t.

metric topology on (X, d) $=$ given topology on X.

$\boxed{\text{Metrizable} \Rightarrow \text{Hausdorff}}$

<u>Metrizable</u> Euclidean top. on $\mathbb{R}^n$, Discrete top. on any set X (disc. metric)

<u>Not metrizable</u> Indisc. top. on X if $|X| \geq 2$, finite comp. top. on X for $|X| = \infty$

(not Hausdorff)

<u>Q</u> Does Hausdorff $\Rightarrow$ Metrizable? No.

Math 190 | Lecture 10 | 1/31/2018

Last Time · A metric on X is a fcn $d: X \times X \to \mathbb{R}_{\geq 0}$

st
- $d(x,y) = 0 \iff x = y$,
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y)$ (Δ inequality)

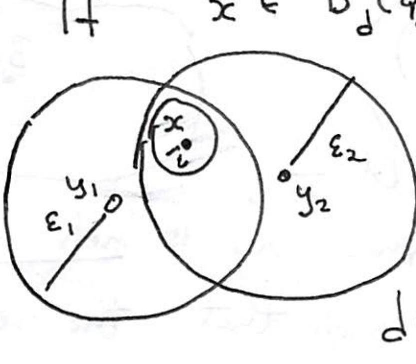
$\forall$ $x,y,z \in X$.

· $B_d(x,\varepsilon) = \{ y \in X : d(x,y) < \varepsilon \}$ "open ball"

Def/Thm Let d be a metric on X. The set $\{ B_d(x,\varepsilon) : x \in X, \varepsilon > 0 \}$ of all open balls forms a basis for a topology on X. This topology is the metric topology on X,

Pf · $x \in B_d(x,1)$, so $\bigcup_{\substack{x \in X \\ \varepsilon > 0}} B_d(x,\varepsilon) = X$.

· If $x \in B_d(y_1, \varepsilon_1) \cap B_d(y_2, \varepsilon_2)$, set



$\varepsilon := \min(\varepsilon_1 - d(x,y_1), \varepsilon_2 - d(x,y_2)) > 0$.

If $d(x,z) < \varepsilon$ then

$$d(z,y_1) \leq d(z,x) + d(x,y_1)$$
$$< \varepsilon + d(x,y_1)$$
$$\leq \varepsilon_1 - d(x,y_1) + d(x,y_1) < \varepsilon_1$$

so $B_d(x,\varepsilon) \subset B_d(y_1,\varepsilon_1)$. Similarly $B_d(x,\varepsilon) \subset B_d(y_2,\varepsilon_2)$.

//

Ex • Metric top on $(\mathbb{R}^n, d_{\text{standard}})$ = Euclidean top. on $\mathbb{R}^n$

• Metric top on $(\mathbb{R}^n, d_{\text{Manhattan}})$ = Euclidean top on $\mathbb{R}^n$

different metrics can generate same topology!

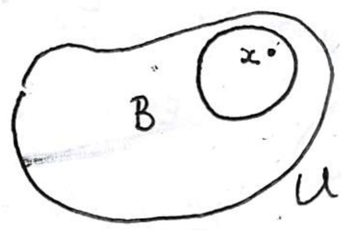
$d_M(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

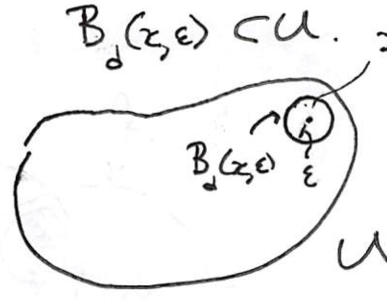
• If d is the discrete metric $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ on X, Metric top on (X,d) = Discrete top. on X

$\left(B_d(x, 1) = \{x\}. \right)$

Rmk Let (X,d) be a metric space (ie d a metric on X, X has metric topology). Then

$U \subset X$ is open $\overset{\text{def}}{\iff}$ for all $x \in U$, $\exists$ some open ball B st $x \in B \subset U$ $\iff$ for all $x \in U$, $\exists\, \varepsilon > 0$ st $B_d(x, \varepsilon) \subset U$.





Def Let X be a topological space. X is metrizable if $\exists$ a metric d on X (the set) such that the metric topology on (X,d) coincides with the given topology on X.

eg discrete top. on X is metrizable (use discrete metric!)

Q// When is X metrizable?

<u>Fact</u> Any metric space X is Hausdorff.

<u>Pf</u> Let $x \neq y \in X$, so $d(x,y) > 0$. Let $\varepsilon = \frac{d(x,y)}{2}$.

Let $U = B_d(x,\varepsilon)$, $V = B_d(y,\varepsilon)$. If $z \in U \cap V$

then $d(x,y) \leq d(x,z) + d(z,y) < \varepsilon + \varepsilon = d(x,y)$. $\divideontimes$

So $U \cap V = \emptyset$. $\blacksquare$

So
- $|X| \geq 2 \Rightarrow$ indisc. top on X not metrizable
- $|X| = \infty \Rightarrow$ finite complement top. on X not metrizable
- Zariski top. on $\mathbb{R}^n$ (or $\mathbb{C}^n$) not metrizable.

(recall $\begin{pmatrix}\text{closed}\\ \text{sets}\end{pmatrix} \Longleftrightarrow \begin{pmatrix}\text{zero loci of}\\ \text{families of polynomials}\end{pmatrix}$)

eg $n=2$, $x - y^2 \longleftrightarrow$

$\leftarrow$ closed set!

<u>Fact</u> Let (X, d_X) & (Y, d_Y) be metric spaces.

Let $f: X \to Y$ be a fcn.

f is CTS $\Longleftrightarrow$ for all $x \in X$, for all $\varepsilon > 0$, $\exists\, \delta > 0$ st $d_X(x,x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$.

(see HW...)

<u>Q</u> Hausdorff $\overset{?}{\Longrightarrow}$ Metrizable $\underline{A}$ <u>NO</u> ...

<u>Fact</u> Let X be a top. space & let $A \subset X$. Let $x \in X$.

If $\exists$ a sequence x_n in A st $x_n \to x$ then $x \in \overline{A}$.

The converse holds if X is metrizable.

Pf Suppose $x_n \to x$ w/ $x_n \in A$ for all n. Let U be a nbhd of x in X. $\exists$ N st $n \geq N \Rightarrow x_n \in U$.

So $x_N \in U \cap A$ & $x \in \overline{A}$.

Suppose $X = (X, d)$ is a metric space & $x \in \overline{A}$.

For all $n \geq 1$, $\exists$ $x_n \in A \cap B_d(x, 1/n)$. We claim that $x_n \to x$. Indeed, if U is a nbhd of x, $\exists$ $\varepsilon > 0$ st $x \in B_d(x, \varepsilon) \subset U$.

If $N > 1/\varepsilon$, we have $n \geq N \Rightarrow$ $x_n \in B_d(x, \varepsilon) \subset U$. So $x_n \to x$. ■

Ex Let S_Ω = minimal uncountable well ordered set, $\overline{S_\Omega} = S_\Omega \cup \{\Omega\}$ ~~maximum~~ largest element

Recall • If $A \subset S_\Omega$ is countable, $\exists$ $x \in S_\Omega$ st $x > a$ for all $a \in A$.

• S_Ω does not have a largest elt
(If M were largest, $\{x \in S_\Omega : x < M\} = S_\Omega - \{M\}$ would be uncountable b/c S_Ω is.)

CLAIM Let X be any ordered set. The order top. on X is Hausdorff.

Pf Let $x < y$ in X. CASE I $\exists$ $c \in X$ st $x < c < y$.
- If $x' < x$ & $y < y'$ then $(x', c) \cap (c, y') = \emptyset$.
- If x is smallest & $y < y'$ then $[x, c) \cap (c, y') = \emptyset$.

- If $x' < x$ & y is <u>largest</u> then $(x', c) \cap (c, y] = \emptyset$.
- If x is <u>smallest</u> & y is <u>largest</u> then $[x, c) \cap (c, y] = \emptyset$.

<u>CASE II</u> $\nexists$ $c \in X$ st $x < c < y$ ... ▨

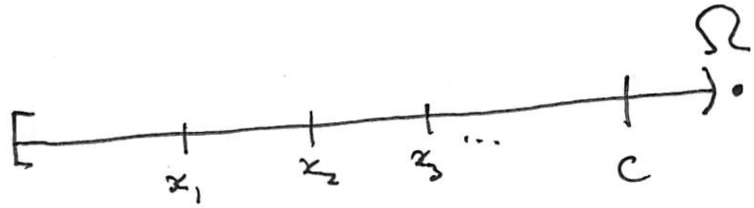
<u>CLAIM</u> Consider the (Hausdorff) order top. $X = \overline{S_\Omega} = S_\Omega \cup \{\Omega\}$.

$\Omega \in$ closure of S_Ω in X,

but $\nexists$ a sequence x_n in S_Ω st $x_n \to \Omega$.

<u>Pf</u> Let $(c, \Omega]$ be any basic open nbhd of Ω (for $c \in S_\Omega$) in X. Since S_Ω does not have a largest element, $(c, \Omega] \neq \{\Omega\}$ & $\Omega \in$ closure of S_Ω in X.

Let x_n be any sequence in S_Ω. Then $\{x_1, x_2, x_3, \dots\} \subset S_\Omega$ is <u>countable</u>, so $\exists$ $c \in S_\Omega$ st $x_n < c$ for all $n \in \mathbb{Z}_{>0}$. Now $(c, \Omega]$ is a nbhd of Ω in X and $x_n \notin (c, \Omega]$ for all $n \geq 1$. Thus $x_n \not\to \Omega$. ▨



S_Ω

<u>COR</u> $\overline{S_\Omega} = S_\Omega \cup \{\Omega\}$ is <u>not</u> metrizable (but it <u>is</u> Hausdorff.)