Math 190: Quotient Topology Supplement

1. Introduction

The purpose of this document is to give an introduction to the *quotient topology*. The quotient topology is one of the most ubiquitous constructions in algebraic, combinatorial, and differential topology. It is also among the most difficult concepts in point-set topology to master. Hopefully these notes will assist you on your journey.

Let X be a topological space. The idea is that we want to glue together points of X to obtain a new topological space. For example, if $X = I^2$ is the unit square, glueing together opposite ends of X (with the same orientation) 'should' produce the torus $S^1 \times S^1$. To encapsulate the (set-theoretic) idea of glueing, let us recall the definition of an *equivalence relation* on a set.

2. Equivalence Relations

Definition 2.1. *Let X be a set. An* equivalence relation $\sim$ *on X is a binary relation on X such that*

- *for all $x \in X$ we have $x \sim x$,*
- *for all $x, y \in X$ we have that $x \sim y$ if and only if $y \sim x$, and*
- *if $x \sim y$ and $y \sim z$, then $x \sim z$ for all $x, y, z \in X$.*

Given $x \in X$, the equivalence class $[x]$ *of X is the subset of X given by*

$$[x] := \{y \in X \,:\, x \sim y\}.$$

We let $X/\sim$ denote the set of all equivalence classes:

$$(X/\sim) := \{[x] \,:\, x \in X\}.$$

Let's look at a few examples of equivalence classes on sets.

Example 2.2. Let $X = \mathbb{R}$ be the set of real numbers. Define a relation $\sim$ on X by $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Check that this is an equivalence relation! Given $x \in \mathbb{R}$, the equivalence class $[x]$ is

$$[x] = \{\dots, x-2, x-1, x, x+1, x+2, \dots\}.$$

Example 2.3. Let X be any set and let $A \subset X$. Define a relation $\sim$ on X by $x \sim y$ if $x = y$ or $x, y \in A$. This is an equivalence relation. The equivalence classes are given by

$$[x] = \begin{cases} A & \text{if } x \in A, \\ \{x\} & \text{if } x \notin A. \end{cases}$$

This equivalence relation 'identifies A to a point'.

Example 2.4. Let $X = [0,1]^2$ be the unit square. We want to identify opposite sides of $[0,1]^2$ while preserving orientation. To do this, we declare

$$\begin{cases} (x,0) \sim (x,1) & \text{for } 0 \leq x \leq 1, \\ (0,y) \sim (1,y) & \text{for } 0 \leq y \leq 1. \end{cases}$$

This declaration generates an equivalence relation on $[0,1]^2$. The equivalence classes are given by

$$\begin{aligned} &[(x,y)] = \{(x,y)\} && \text{for } 0 < x, y < 1, \\ &[(x,0)] = \{(x,0),(x,1)\} && \text{for } 0 < x < 1, \\ &[0,y)] = \{(0,y),(1,y)\} && \text{for } 0 < y < 1, \\ &[(0,0)] = \{(0,0),(0,1),(1,0),(1,1)\}. \end{aligned}$$

Pictorially, the points in the interior of the square are singleton equivalence classes, the points on the edges get identified, and the four corners of the square are identified.

Recall that on the first day of class I talked about glueing sides of $[0,1]^2$ together to get geometric objects (cylinder, torus, Möbius strip, Klein bottle, real projective space). What are the equivalence relations and equivalence classes for these identifications? (The last example handled the case of the torus.)

One final remark about equivalence relations. Let X be any set and let $\sim$ be any equivalence relation on X. We have a canonical surjective map $\pi : X \to X/\sim$ defined by

$$\pi : x \mapsto [x].$$

That is, π sends any element to its equivalence class.

3. The Quotient Topology: Definition

Thus far we've only talked about *sets*. We want to talk about *spaces*. Let X be a topological space and let $\sim$ be an equivalence relation on X. Then $(X/\sim)$ is a *set* of equivalence classes. We want to topologize this set in a fashion consistent with our intuition of glueing together points of X. The gadget for doing this is as follows.

Definition 3.1. *Let X be a topological space, let $\sim$ be an equivalence relation on X, and let $X/\sim$ be the corresponding set of equivalence classes. The* quotient topology *on $X/\sim$ has open sets defined as follows. We declare $U \subset (X/\sim)$ to be open if and only if the union*

$$\bigcup_{[x] \in U} [x] \subset X$$

is open in X.

(Observe that U is a collection of equivalence classes of X (and in particular a collection of subsets of X), so that the above union makes sense and really is a subset of X.)

Our first task is to show that this is actually a topology on $X/\sim$. This boils down to some set theoretic facts about equivalence classes.

Proposition 3.2. *The definition above gives a topology on $X/\sim$.*

Proof. We have that

$$\bigcup_{[x]\in(X/\sim)} [x] = X,$$

which is open in X, so that $X/\sim$ is open in $X/\sim$. Moreover, we have that

$$\bigcup_{[x]\in\emptyset} [x] = \emptyset,$$

which is open in X, so that $\emptyset$ is open in $X/\sim$.

Let $\{U_\alpha\}$ be an arbitrary nonempty collection of open sets in $X/\sim$. Then

$$\bigcup_{[x]\in\bigcup_\alpha U_\alpha} [x] = \bigcup_\alpha \left(\bigcup_{x\in U_\alpha} [x] \right).$$

For any α, we have that $\bigcup_{[x]\in U_\alpha} [x]$ is open in X. Since X is a topological space, we conclude that $\bigcup_\alpha \left(\bigcup_{x\in U_\alpha} [x]\right)$ is also open in X. Therefore, we have that $\bigcup_\alpha U_\alpha$ is open in $X/\sim$.

Let $\{U_1, \dots, U_n\}$ be a finite collection of open sets in $X/\sim$. Then

$$\bigcup_{x\in\bigcap_{i=1}^n U_i} [x] = \bigcap_{i=1}^n \left(\bigcup_{x\in U_i} [x] \right).$$

For all $1 \leq i \leq n$, we have that $\bigcup_{[x]\in U_i} [x]$ is open in X. Since X is a topological space, we conclude that $\bigcap_{i=1}^n \left(\bigcup_{x\in U_i} [x]\right)$ is also open in X. Therefore, we have that $\bigcap_{i=1}^n U_i$ is open in $X/\sim$. We conclude that the collection of open sets defined above actually gives a topology on $X/\sim$. □

At this point, the quotient topology is a somewhat mysterious object. Just knowing the open sets in a topological space can make the space itself seem rather inscrutable. However, we can prove the following result about the canonical map $\pi : X \to X/\sim$ introduced in the last section.

Proposition 3.3. *Let X be a topological space and let $\sim$ be an equivalence relation on X. Endow the set $X/\sim$ with the quotient topology. The canonical surjection $\pi : X \to X/\sim$ given by $\pi : x \mapsto [x]$ is continuous.*

Proof. Let $U \subset X/\sim$ be an open set. Then U is a union of equivalence classes in $X/\sim$ such that the union $\bigcup_{[x]\in U}[x]$ is open in X. But we have $\pi^{-1}(U) = \bigcup_{[x]\in U}[x]$. □

The proof of the last proposition was easy. The definition of the quotient topology made it so.

4. Functions on Equivalence Classes

Okay, so we have a topology on $X/\sim$. However, this is not enough for our purposes. We want to (for example) identify the quotients coming from identifying opposite sides of $[0,1]^2$ with the cylinder, the torus, etc. In general, we want an effective way to prove that a given (at this point mysterious) quotient $X/\sim$ is homeomorphic to a (known and loved) topological space Y. This means that we need to find mutually inverse continuous maps from $X/\sim$ to Y and vice versa. In particular, we need to construct a *function* $f : (X/\sim) \to Y$. Let's talk about this first on the level of *sets*.

If one wants to define a function $f : (X/\sim) \to Y$, one will typically write down some formula for $f([x])$ based on a representative x for the equivalence class $[x]$. For example, let $X = \mathbb{R}$ and define $\sim$ on X by $x \sim y$ if and only if $y - x \in \mathbb{Z}$. We could try to define $f : (\mathbb{R}/\sim) \to S^1$ by

$$f([x]) = e^{\pi i x}.$$

But there is a problem here. Since $0 \sim 1$, we have that $[0] = [1]$. However the right hand side of the above formula gives $e^{\pi i * 0} = 1 \neq -1 = e^{\pi i * 1}$. Therefore, the above "function" is not well-defined. On the other hand, we could define $g : (\mathbb{R}/\sim) \to S^1$ by

$$g([x]) = e^{4\pi i x}.$$

Then if $x \sim y$, we have that $y - x \in \mathbb{Z}$, so that

$$e^{4\pi i x} = 1 * e^{4\pi i x} = e^{4\pi i(y-x)}e^{4\pi i x} = e^{4\pi i y}.$$

It follows that g is a well-defined function.

The following proposition, whose proof is standard, encapsulates the ideas in the last paragraph.

Proposition 4.1. *Let X and Y be sets and let $\sim$ be an equivalence relation on X. Let $f : X \to Y$ be a function with the property that $f(x) = f(x')$ whenever $x \sim x'$ in X. Let $\pi : X \to X/\sim$ be the canonical surjection*

$$\pi : x \mapsto [x]$$

which sends every element to its equivalence class. There exists a unique function $\overline{f} : (X/\sim) \to Y$ satisfying

$$f = \overline{f} \circ \pi.$$

Moreover, we have that

$$\overline{f}([x]) = f(x)$$

for any $[x] \in (X/\sim)$*.*

Proof. If such a function $\overline{f}$ exists, the relation $f = \overline{f} \circ \pi$ forces

$$f(x) = \overline{f} \circ \pi(x) = \overline{f}([x])$$

for any $x \in X$. The uniqueness of $\overline{f}$ follows.

For the existence of $\overline{f}$, given $[x] \in (X/\sim)$, let x' be any representative of $[x]$. Set $\overline{f}([x]) := f(x')$. The assumption on f makes this choice independent of the choice of x'. □

5. The Universal Property of the Quotient Topology

It's time to boost the material in the last section from sets to topological spaces. The following result is the most important tool for working with quotient topologies.

Theorem 5.1. (The Universal Property of the Quotient Topology) *Let* X *be a topological space and let* $\sim$ *be an equivalence relation on* X*. Endow the set* $X/\sim$ *with the quotient topology and let* $\pi : X \to X/\sim$ *be the canonical surjection.*

Let Y *be another topological space and let* $f : X \to Y$ *be a continuous function with the property that* $f(x) = f(x')$ *whenever* $x \sim x'$ *in* X*. There exists a unique continuous function* $\overline{f} : (X/\sim) \to Y$ *such that*

$$f = \overline{f} \circ \pi.$$

Proof. We already know that there exists a unique *function* $\overline{f} : (X/\sim) \to Y$ such that $f = \overline{f} \circ \pi$ and that this function is given by the formula $\overline{f}([x]) = f(x)$. We only need to check that $\overline{f}$ is actually continuous.

Let $U \subset Y$ be an open set. Since f is continuous, we know that $f^{-1}(U) \subset X$ is open. Since $f(x) = f(x')$ whenever $x \sim x'$ in X, we also know that $f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} [x]$ is a union of equivalence classes in X. Since

$$\bigcup_{[x] \in \overline{f}^{-1}(U)} [x] = \bigcup_{x \in f^{-1}(U)} [x] = f^{-1}(U)$$

is an open subset of X, it follows that $\overline{f}^{-1}(U)$ is an open subset of $X/\sim$. We conclude that $\overline{f}$ is a continuous function. □

The name 'Universal Property' stems from the following exercise. Morally, it says that the behavior with respect to maps described above completely characterizes the quotient topology on $X/\sim$ (or, more correctly, the triple $(X, X/\sim, \pi)$ where $\pi : X \to X/\sim$ is the canonical projection).

Problem 5.2. *Let X be a topological space, let $\sim$ be an equivalence relation on X, and let $\mathcal{T}$ be a topology on the set $X/\sim$ which satisfies the following property.*

> *The canonical surjection $\pi : X \to X/\sim$ is continuous (with respect to $\mathcal{T}$).*
>
> *Moreover, for any topological space Y and any continuous map $f : X \to Y$ such that $f(x) = f(x')$ whenever $x \sim x'$ in X, there exists a unique continuous map $\overline{f} : (X/\sim) \to Y$ such that $f = \overline{f} \circ \pi$.*

Then $\mathcal{T}$ is the quotient topology on $X/\sim$.

Alright, how does this actually work in practice? Let's consider the following problem.

Problem 5.3. *Endow $X = \mathbb{R}$ with the standard topology. Define an equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Prove that $\mathbb{R}/\sim$ is homeomorphic to $Y = S^1$ (with the standard topology).*

This problem is a typical situation. We have a space that is known and loved (namely $\mathbb{R}$), we have an equivalence relation on it, and we have to show that the resulting quotient is homeomorphic to another known and loved space (namely S^1). Here is the basic program for doing this.

(1) Find a candidate continuous function $\tilde{f} : X \to Y$.
(2) Prove that $\tilde{f}(x) = \tilde{f}(x')$ whenever $x \sim x'$ in X. Then the function $f : (X/\sim) \to Y$ defined by

$$f([x]) = \tilde{f}(x)$$

is well defined. The Universal Property of the Quotient Topology guarantees that f is continuous.
(3) Find a candidiate inverse continuous function $g : Y \to (X/\sim)$.
(4) Prove that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

Here is how we perform this task in the context of the above problem. (As usual, we identify $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ with a subset of the complex plane.

(1) Define $\tilde{f} : \mathbb{R} \to S^1$ by

$$\tilde{f}(t) = e^{2\pi i t}.$$

Then $\tilde{f}$ is continuous (from our knowledge of calculus) as a map between Euclidean spaces.
(2) Suppose $t \sim t'$ in $\mathbb{R}$. Then $t' - t \in \mathbb{Z}$. It follows that

$$\tilde{f}(t) = e^{2\pi i t} = e^{2\pi i (t'-t)} e^{2\pi i t} = e^{2\pi i t'} = \tilde{f}(t').$$

Therefore, the induced map $f : (\mathbb{R}/\sim) \to S^1$ given by

$$f([t]) = e^{2\pi i t}$$

is well-defined and continuous.

(3) We define two subsets of S^1 as follows:

$$A_1 := \{z \in S^1 \,:\, \Im(z) \geq 0\}$$
$$A_2 := \{z \in S^1 \,:\, \Im(z) \leq 0\}.$$

(Here $\Im(z)$ denotes the imaginary part of a complex number z.) Observe that both A_1 and A_2 are closed in S^1 and that $S^1 = A_1 \cup A_2$. Moreover, we have that $A_1 \cap A_2 = \{\pm 1\}$. We define functions $\tilde{g}_i : A_i \to \mathbb{R}$ for $i = 1, 2$ as follows.

Given $z \in A_1$, we can uniquely write $z = e^{2\pi i t}$ for $0 \leq t \leq \frac{1}{2}$. Define $\tilde{g}_1 : A_1 \to \mathbb{R}$ by $\tilde{g}_1(z) = t$. Given $z \in A_2$, we can uniquely write $z = e^{2\pi i t'}$ for some $\frac{1}{2} \leq t' \leq 1$. Define $\tilde{g}_2 : A_2 \to \mathbb{R}$ by $\tilde{g}_2(z) = t'$. Then $\tilde{g}_1$ and $\tilde{g}_2$ are continuous functions by our knowledge of calculus. (N.B.: The functions $\tilde{g}_1$ and $\tilde{g}_2$ do *not* agree on $A_1 \cap A_2$, so do not paste to give a well defined function $S^1 \to \mathbb{R}$.)

Now define functions $g_i : A_i \to (\mathbb{R}/\sim)$ for $i = 1, 2$ by $g_i = \pi \circ \tilde{g}_i$. Since π is continuous and $\tilde{g}_i$ is continuous, we have that g_i is continuous. Moreover, we have that $g_1(-1) = [1/2] = g_2(-1)$ and $g_1(1) = [0] = [1] = g_2(1)$. Since g_1 and g_2 agree on $A_1 \cap A_2$, the Pasting Lemma implies that the function $g : S^1 \to (\mathbb{R}/\sim)$ defined by $g|_{A_1} = g_1$ and $g|_{A_2} = g_2$ is well defined and continuous. Observe that $g(e^{2\pi i t}) = [t]$ for any $t \in \mathbb{R}$ (as can be checked on A_1 and A_2).

(4) Let $t \in \mathbb{R}$, so that $[t] \in \mathbb{R}/\sim$. We calculate

$$g \circ f([t]) = g(e^{2\pi i t}) = [t].$$

Now let $z \in S^1$ and write $z = e^{2\pi i t}$. We calculate

$$f \circ g(z) = f \circ g(e^{2\pi i t}) = f([t]) = e^{2\pi i t} = z.$$

It follows that f and g are mutually inverse continuous maps, so that $\mathbb{R}/\sim$ and S^1 are homeomorphic.

Let's observe what just happened (for it is fairly typical).

In Step 1, we define a candidate function $\tilde{f} : X \to Y$ out of the 'numerator' space X and into the target space Y. The continuity of $\tilde{f}$ will typically be obvious from calculus.

In Step 2 we check that $\tilde{f}$ is constant on equivalence classes – a purely set theoretic check! We appeal to the Universal Property to get that the induced function $f : (X/\sim) \to Y$ is continuous.

Step 3 typically requires the most ingenuity – the Pasting Lemma is not infrequently applied here. Probably the most useful fact is that the quotient map $X \to (X/\sim)$ is continuous.

Step 4 is another set-theoretic check that the candidate inverse maps are actually inverse to one another.

6. Quotient Maps

There is another way to introduce the quotient topology in terms of so-called 'quotient maps'.

Definition 6.1. *Let X and Y be topological spaces. A map $g : X \to Y$ is a* quotient map *if g is surjective and for any set $U \subset Y$ we have that U is open in Y if and only if $g^{-1}(U)$ is open in X.*

In particular, quotient maps are continuous. Moreover, if $g : X \to Y$ is a quotient map, the topology on Y is completely determined by the function g and the topology on X.

In general, it is not especially easy to identify quotient maps – even between Euclidean spaces. For example, consider the subspace $X \subset \mathbb{R}^2$ given by

$$X = \{(x, y) \in \mathbb{R}^2 \,:\, xy = 1\} \cup \{(0, 0)\}.$$

Define $g : X \to \mathbb{R}$ by $g(x, y) = x$. Then g is surjective and continuous. However, we have that $g^{-1}(\{0\}) = \{(0, 0)\}$, which is open in X. However, $\{0\}$ is not open in $\mathbb{R}$. In particular, unlike the situation with continuous maps, you can't just say that a map between Euclidean spaces is a quotient map from knowledge of calculus.

The relationship between quotient maps and the quotient topology is as follows. The proof is a tautology.

Proposition 6.2. *Let X be a topological space and let A be a set. Let $g : X \to A$ be a surjective function. Define an equivalence relation $\sim$ on X by $x \sim x'$ if and only if $g(x) = g(x')$. Then we can identify $X/\sim$ with A via $[x] \mapsto g(x)$. The quotient topology on $X/\sim$ is the unique topology on $X/\sim$ which turns g into a quotient map.*

Here is a criterion which is often useful for checking whether a given map is a quotient map. If $f : A \to B$ is a map of sets, let us call a subset $V \subset A$ *saturated* (with respect to f) if whenever $a \in V$ and $f(a) = f(a')$, we have that $a' \in V$.

Proposition 6.3. *Let X and Y be topological spaces and let $f : X \to Y$ be a surjective continuous function. Then f is a quotient map if and only if for every* saturated *open set $V \subset X$, we have that $f(V) \subset Y$ is open.*

7. Problems

Problem 7.1. *Let $X = D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the closed unit disc (in the standard topology). Identify S^1 with the boundary of D^2. Define an equivalence relation $\sim$ on D^2 by declaring any two points on S^1 to be equivalent. The resulting quotient space $D^2/\sim$ is denoted D^2/S^1 (in slang, we're identifying S^1 to a point). Prove that D^2/S^1 is homeomorphic to S^2.*

Problem 7.2. *Let $X = D^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$ be the closed unit ball. Define an equivalence relation $\sim$ on X by declaring $x \sim -x$ if x lies on the boundary S^{n-1} of D^n. Prove that $D^n/\sim$ is homeomorphic to the projective space P^n.*

Problem 7.3. *Let D_1 and D_2 be two copies of the 2-dimensional disc D^2. Prove that the quotient space obtained by identifying the boundaries of D_1 and D_2 is homeomorphic to S^2.*

Problem 7.4. *Let D^2 be the 2-dimensional disc and let M be the Möbius strip. Prove that the quotient space obtained by identifying the boundary circles of D^2 and M is homeomorphic to the projective space P^2.*

Problem 7.5. *Let M_1 and M_2 be two copies of the Möbius strip. Prove that the quotient space obtained by identifying the boundary circles of M_1 and M_2 is homeomorphic to the Klein bottle K.*

Problem 7.6. *Define an equivalence relation $\sim$ on $\mathbb{R}^n$ by $x \sim y$ if and only if $x - y \in \mathbb{Z}^n$. Prove that $\mathbb{R}^n/\sim$ is homeomorphic to the n-dimensional torus $S^1 \times \cdots \times S^1$ (n copies of S^1).*

Problem 7.7. *Prove that $\mathbb{R}/\mathbb{Q}$ is not Hausdorff. (This is the quotient space obtained by starting with $\mathbb{R}$ and then identifying $\mathbb{Q}$ to a single point.)*

Problem 7.8. *Let $\mathbb{R}_1$ and $\mathbb{R}_2$ be two copies of the real line $\mathbb{R}$. For a real number x, let $x_1 \in \mathbb{R}_1$ and $x_2 \in \mathbb{R}_2$ be the two copies of x. Define an equivalence relation on the disjoint union $\mathbb{R}_1 \amalg \mathbb{R}_2$ by $x_1 \sim x_2$ for all real $x \neq 0$. Let $X = (\mathbb{R}_1 \amalg \mathbb{R}_2)/\sim$ be the quotient space. (X is called the 'line with two origins'.) Prove that X is not Hausdorff, but that every point in X has a neighborhood homeomorphic to $\mathbb{R}$.*