**Math 200A**
**Fall 2015**
**Homework 5**
**Due 11/6/2015**

**Problem 1:** Let $R$ be a commutative ring with 1. Recall that an element $x \in R$ is *nilpotent* if $x^n = 0$ for some $n > 0$.

(1) Prove that the set $\mathfrak{N}$ of nilpotent elements of $R$ forms an ideal in $R$.
(2) Prove that $\mathfrak{N}$ is the intersection of all of the prime ideals in $R$.

$\mathfrak{N}$ is called the *nilradical* of $R$. (Hint: For Part 2, if $x \in R$ is not nilpotent, let $X = \{1, x, x^2, \dots\}$ and consider the collection of all ideals $I$ which satisfy $I \cap X = \emptyset$. By Zorn's Lemma, this collection has a maximal element $M$ under inclusion. Prove that $M$ is prime.)

**Problem 2:** Let $F$ be a field and consider the ring $M_n(F)$ of $n \times n$ matrices with entries in $F$. Prove that $M_n(F)$ does not contain any nontrivial ideals.

**Problem 3:** Let $R$ be a commutative ring. A *formal power series* (in the variable $x$) with coefficients in $R$ is a formal expression

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

where $a_0, a_1, a_2 \cdots \in R$. The set $R[[x]]$ of all formal power series forms a commutative ring under the standard addition and multiplication rules.

(1) Prove that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in $R[[x]]$ if and only if $a_0$ is a unit in $R$.
(2) Prove that if $R$ is an integral domain, then $R[[x]]$ is an integral domain.
(3) If $F$ is a field, show that the set

$$M := \left\{ \sum_{n=0}^{\infty} a_n x^n \ : \ a_0 = 0 \right\}$$

of all formal power series in $F[[x]]$ with vanishing constant term is the unique maximal ideal of $F[[x]]$. (Rings with unique maximal ideals are called *local rings*.)

**Problem 4:** Let $F$ be a field and let $R = F[x_1, x_2, \dots]$ be the polynomial ring over $F$ in infinitely many variables $x_1, x_2, \dots$. Prove that $R$ contains an ideal which is not finitely generated.

**Problem 5:** Let $F$ be a field and consider the subring $R = F[x, xy, xy^2, xy^3, \dots]$ of $F[x, y]$ which is generated by $x, xy, xy^2, xy^3, \dots$. Prove that $R$ contains an ideal which is not finitely generated.

**Problem 6:** (Optional - A "group algebra" for an infinite group.)

(1) Let $G$ be an infinite group and consider the problem of trying to define the group algebra of $G$ over $\mathbb{R}$. Let $A$ be the set of all (possibly infinite) expressions of

the form

$$\sum_{g \in G} a_g g$$

for $a_g \in \mathbb{R}$. Prove that $A$ is not necessarily a ring with respect to componentwise addition and the same multiplication formula that we used to define $\mathbb{R}G$ for $G$ finite. (One way to get around this problem is to consider only sums of the above kind where $a_g = 0$ for all but finitely many $g \in G$. The next part shows a "better" way in some cases.)

(2) Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the circle (a very infinite group under multiplication). A given real-valued function $f : S^1 \to \mathbb{R}$ is called *square integrable* if $\int_{S^1} f^2$ exists and is finite. (You may assume any plausible property of $\int_{S^1}$ that you want for this problem!) The set of all (measurable) square integrable functions $S^1 \to \mathbb{R}$ is denoted $L^2(S^1)$. Show that $L^2(S^1)$ is a ring under pointwise addition and convolution multiplication:

$$(f * g)(z) := \int_{y \in S^1} f(y) g(y^{-1} z).$$

(If $G$ is a 'compact Lie group' (the circle is an example thereof), we can define $\int_G f$ for nice functions $f : G \to \mathbb{R}$ using 'Haar measure' on $G$. The above formulas define a ring structure on $L^2(G)$. This is the 'correct' analytical definition of the group algebra on $G$ and plays a very important role in Lie theory.)

(3) Explain why the convolution product in Part 2 is 'the same' as the product in the group algebra $\mathbb{R}G$ for $G$ finite. (Hint: Think about elements of $\mathbb{R}G$ as functions $G \to \mathbb{R}$. Since $G$ is finite, integrals are sums and convergence doesn't pose any problems!)