Problem 1: Let $R$ be a commutative ring with 1. Recall that an element $x \in R$ is \emph{nilpotent} if $x^n = 0$ for some $n > 0$.

1. Prove that the set $\mathfrak{N}$ of nilpotent elements of $R$ forms an ideal in $R$.
2. Prove that $\mathfrak{N}$ is the intersection of all of the prime ideals in $R$.

$\mathfrak{N}$ is called the \emph{nilradical} of $R$. (Hint: For Part 2, if $x \in R$ is not nilpotent, let $X = \{1, x, x^2, \ldots \}$ and consider the collection of all ideals $I$ which satisfy $I \cap X = \emptyset$. By Zorn’s Lemma, this collection has a maximal element $M$ under inclusion. Prove that $M$ is prime.)

Problem 2: Let $F$ be a field and consider the ring $M_n(F)$ of $n \times n$ matrices with entries in $F$. Prove that $M_n(F)$ does not contain any nontrivial ideals.

Problem 3: Let $R$ be a commutative ring. A \emph{formal power series} (in the variable $x$) with coefficients in $R$ is a formal expression

$$a_0 + a_1x + a_2x^2 + \cdots$$

where $a_0, a_1, a_2 \cdots \in R$. The set $R[[x]]$ of all formal power series forms a commutative ring under the standard addition and multiplication rules.

1. Prove that $\sum_{n=0}^{\infty} a_nx^n$ is a unit in $R[[x]]$ if and only if $a_0$ is a unit in $R$.
2. Prove that if $R$ is an integral domain, then $R[[x]]$ is an integral domain.
3. If $F$ is a field, show that the set

$$M := \left\{ \sum_{n=0}^{\infty} a_nx^n : a_0 = 0 \right\}$$

of all formal power series in $F[[x]]$ with vanishing constant term is the unique maximal ideal of $F[[x]]$. (Rings with unique maximal ideals are called \emph{local rings}.)

Problem 4: Let $F$ be a field and let $R = F[x_1, x_2, \ldots]$ be the polynomial ring over $F$ in infinitely many variables $x_1, x_2, \ldots$. Prove that $R$ contains an ideal which is not finitely generated.

Problem 5: Let $F$ be a field and consider the subring $R = F[x, xy, xy^2, xy^3, \ldots]$ of $F[x, y]$ which is generated by $x, xy, xy^2, xy^3, \ldots$. Prove that $R$ contains an ideal which is not finitely generated.

Problem 6: (Optional - A “group algebra” for an infinite group.)

1. Let $G$ be an infinite group and consider the problem of trying to define the group algebra of $G$ over $\mathbb{R}$. Let $A$ be the set of all (possibly infinite) expressions of
the form

\[
\sum_{g \in G} a_g g
\]

for \(a_g \in \mathbb{R}\). Prove that \(A\) is not necessarily a ring with respect to componentwise addition and the same multiplication formula that we used to define \(\mathbb{R}G\) for \(G\) finite. (One way to get around this problem is to consider only sums of the above kind where \(a_g = 0\) for all but finitely many \(g \in G\). The next part shows a "better" way in some cases.)

(2) Let \(S^1 = \{z \in \mathbb{C} : |z| = 1\}\) be the circle (a very infinite group under multiplication). A given real-valued function \(f : S^1 \to \mathbb{R}\) is called \textit{square integrable} if \(\int_{S^1} f^2\) exists and is finite. (You may assume any plausible property of \(\int_{S^1}\) that you want for this problem!) The set of all (measurable) square integrable functions \(S^1 \to \mathbb{R}\) is denoted \(L^2(S^1)\). Show that \(L^2(S^1)\) is a ring under pointwise addition and convolution multiplication:

\[(f * g)(z) := \int_{y \in S^1} f(y) g(y^{-1}z).\]

(If \(G\) is a ‘compact Lie group’ (the circle is an example thereof), we can define \(\int_G f\) for nice functions \(f : G \to \mathbb{R}\) using ‘Haar measure’ on \(G\). The above formulas define a ring structure on \(L^2(G)\). This is the ‘correct’ analytical definition of the group algebra on \(G\) and plays a very important role in Lie theory.)

(3) Explain why the convolution product in Part 2 is ‘the same’ as the product in the group algebra \(\mathbb{R}G\) for \(G\) finite. (Hint: Think about elements of \(\mathbb{R}G\) as functions \(G \to \mathbb{R}\). Since \(G\) is finite, integrals are sums and convergence doesn’t pose any problems!)