Math 202B: Winter 2015
Homework 5
Due 2/13/2015

Problem 1: Let $\lambda \vdash n$. We proved in class that the set of standard polytabloids $\{e_T : T \in SYT(\lambda)\}$ forms a basis for the Young module S^λ. (The representing matrices for permutations in $\mathfrak{S}_n$ with respect to this basis give *Young's natural representation.*) When $\lambda = (2,2)$, find the representing matrices for the adjacent transpositions $(1,2),(2,3),(3,4) \in \mathfrak{S}_4$ acting on $S^{(2,2)}$ with respect to this basis.

Problem 2: Let $\lambda \vdash n$ and let T be a fixed λ-tableau. The *Young idempotent* $\epsilon_T \in \mathbb{C}[\mathfrak{S}_n]$ is the group algebra element

$$\epsilon_T := R_T^+ C_T^-,$$

where R_T and C_T are the row and column stabilizers of T. The left ideal $\mathbb{C}[\mathfrak{S}_n]\epsilon_T$ in the symmetric group algebra carries an action of $\mathfrak{S}_n$ given by left multiplication.

(1) Prove that the $\mathfrak{S}_n$-module $\mathbb{C}[\mathfrak{S}_n]\epsilon_T$ is isomorphic to S^λ (for any λ-tableau T).
(2) Let $\lambda_1 = (3), \lambda_2 = (2,1), \lambda_3 = (1,1,1)$ be the three partitions of 3. Let T_1, T_2, and T_3 be your favorite λ_1, λ_2, and λ_3-tableaux. Calculate the products $\epsilon_{T_i}\epsilon_{T_j} \in \mathbb{C}[\mathfrak{S}_3]$ for $1 \leq i, j \leq 3$.

Problem 3: Let $\lambda, \mu \vdash n$. The *Kronecker product* of the $\mathfrak{S}_n$-irreducibles S^λ and S^μ is the $\mathfrak{S}_n$-representation $S^\lambda \otimes S^\mu$ and the *Kronecker coefficients* $g(\lambda,\mu,\nu)$ are the nonnegative integers defined by

$$S^\lambda \otimes S^\mu \cong_{\mathfrak{S}_n} \bigoplus_{\nu \vdash n} g(\lambda,\mu,\nu) S^\nu.$$

Calculate the numbers $g((3,1),(2,2),\nu)$ for all $\nu \vdash 4$. The problem of finding $g(\lambda,\mu,\nu)$ in general is known as the *Kronecker problem* and is famously difficult – no manifestly positive formula for $g(\lambda,\mu,\nu)$ is known in general.

Problem 4: Prove that every irreducible character of $\mathfrak{S}_n$ is an integer valued function.

Problem 5: Prove that, up to sign, the determinant of the character table of $\mathfrak{S}_n$ is

$$\prod_{\lambda \vdash n} \prod_{\lambda_i \in \lambda} \lambda_i.$$

Problem 6: Let $\lambda = (4,4,3,1) \vdash 12$. Calculate the decomposition of the restriction $S^\lambda \downarrow_{\mathfrak{S}_9}$ into $\mathfrak{S}_9$-irreducibles. What about the induction $S^\lambda \uparrow^{\mathfrak{S}_{15}}$?

Problem 7: If k is any field, we can still define the $k[\mathfrak{S}_n]$-modules M^λ and S^λ for $\lambda \vdash n$ by taking coefficients in k. Give an example to show that when k has positive characteristic, the $k[\mathfrak{S}_n]$-module S^λ is not necessarily irreducible.