Math 202B: Winter 2017
Homework 2
Due 1/25/2017

Problem 1: Let k be a field of characteristic 0 and endow the vector space $V = k^n$ with the defining representation of $\mathfrak{S}_n$. Describe the structure of the endomorphism algebra $\mathrm{End}_{\mathfrak{S}_n}(V)$.

Problem 2: Let $D_n = \langle r, s \mid r^n = s^2 = rsrs = e \rangle$ be the dihedral group of symmetries of a regular n-gon (we can think of r as rotation though an angle $2\pi/n$ and s as a distinguished reflection). Determine the degree 1 representations of D_n.

Problem 3: Let G be a finite group and let V and W be finite-dimensional G-modules over $\mathbb{C}$. Find a formula for the dimension of the vector space $\mathrm{Hom}_G(V, W)$ in terms of the decompositions of V and W into irreducible representations.

Problem 4: Let G be a finite group and let V be an irreducible finite-dimensional G-module over $\mathbb{C}$. Let $K \subseteq G$ be a conjugacy class and define a linear operator $\varphi : V \to V$ by the formula $\varphi(v) := \sum_{g \in K} g.v$. Prove that there exists a scalar $c \in \mathbb{C}$ such that $\varphi(v) = cv$ for all $v \in V$.

Problem 5: Let k be any field. Prove that the center of the k-algebra $\mathrm{Mat}_d(k)$ of $d \times d$ matrices with entries in k is $\{cI_d \,:\, c \in k\}$.

Problem 6: Let k be an algebraically closed field of characteristic 0 and let A be a k-algebra. The *opposite algebra* A^{op} is a copy of the k-vector space A, but with multiplication reversed:

$$a \cdot_{op} b = ba,$$

for all $a, b \in A$.

(1) Let A be the k-algebra of $d \times d$ matrices with entries in k. Prove that A and A^{op} are isomorphic as algebras.

(2) Let G be a finite group and let $k[G]$ be its left regular representation with endomorphism algebra $\mathrm{End}_G(k[G])$. Describe an algebra isomorphism

$$\varphi : k[G] \to \mathrm{End}_G(k[G])^{op}.$$

(3) Prove that we have an isomorphism of matrix algebras

$$k[G] \cong \mathrm{Mat}_{m_1}(k) \oplus \cdots \oplus \mathrm{Mat}_{m_r}(k)$$

for some integers $m_1, \ldots, m_r$.

This is a special case of the Artin-Wedderburn decomposition theorem. In general, if A is any finite-dimensional algebra over an algebraically closed field k whose radical $\mathrm{rad}(A)$ satisfies $\mathrm{rad}(A) = 0$, then A is isomorphic to a direct sum of matrix algebras over k.