Math 202B: Winter 2017
Homework 2
Due 1/25/2017

Problem 1: Let $k$ be a field of characteristic 0 and endow the vector space $V = k^n$ with the defining representation of $S_n$. Describe the structure of the endomorphism algebra $\text{End}_{S_n}(V)$.

Problem 2: Let $D_n = \langle r, s \mid r^n = s^2 = rsrs = e \rangle$ be the dihedral group of symmetries of a regular $n$-gon (we can think of $r$ as rotation though an angle $2\pi/n$ and $s$ as a distinguished reflection). Determine the degree 1 representations of $D_n$.

Problem 3: Let $G$ be a finite group and let $V$ and $W$ be finite-dimensional $G$-modules over $\mathbb{C}$. Find a formula for the dimension of the vector space $\text{Hom}_G(V, W)$ in terms of the decompositions of $V$ and $W$ into irreducible representations.

Problem 4: Let $G$ be a finite group and let $V$ be an irreducible finite-dimensional $G$-module over $\mathbb{C}$. Let $K \subseteq G$ be a conjugacy class and define a linear operator $\varphi : V \to V$ by the formula $\varphi(v) := \sum_{g \in K} g.v$. Prove that there exists a scalar $c \in \mathbb{C}$ such that $\varphi(v) = cv$ for all $v \in V$.

Problem 5: Let $k$ be any field. Prove that the center of the $k$-algebra $\text{Mat}_d(k)$ of $d \times d$ matrices with entries in $k$ is $\{cI_d : c \in k\}$.

Problem 6: Let $k$ be an algebraically closed field of characteristic 0 and let $A$ be a $k$-algebra. The opposite algebra $A^{op}$ is a copy of the $k$-vector space $A$, but with multiplication reversed: $a \cdot_{op} b = ba$, for all $a, b \in A$.

(1) Let $A$ be the $k$-algebra of $d \times d$ matrices with entries in $k$. Prove that $A$ and $A^{op}$ are isomorphic as algebras.

(2) Let $G$ be a finite group and let $k[G]$ be its left regular representation with endomorphism algebra $\text{End}_G(k[G])$. Describe an algebra isomorphism $\varphi : k[G] \to \text{End}_G(k[G])^{op}$.

(3) Prove that we have an isomorphism of matrix algebras $k[G] \cong \text{Mat}_{m_1}(k) \oplus \cdots \oplus \text{Mat}_{m_r}(k)$ for some integers $m_1, \ldots, m_r$.

This is a special case of the Artin-Wedderburn decomposition theorem. In general, if $A$ is any finite-dimensional algebra over an algebraically closed field $k$ whose radical $\text{rad}(A)$ satisfies $\text{rad}(A) = 0$, then $A$ is isomorphic to a direct sum of matrix algebras over $k$. 

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