Math 202B: Winter 2017
Homework 6
Due 3/13/2017

Problem 1: (202 Qual, S 2013)
(1) Consider the Young subgroup \( S_{(3,2,1)} \) of \( S_6 \). Let \( T \) be the trivial representation of \( S_{(3,2,1)} \) and let \( A \) be the sign (i.e., alternating) representation of \( S_{(3,2,1)} \). Calculate the decomposition of the induced modules \( T \uparrow_{S_6} \) and \( A \uparrow_{S_6} \) as direct sums of irreducible \( S_6 \)-modules.
(2) Find the decomposition of the Kronecker product
\[ S_{(4,1)} \otimes S_{(2,2,1)} \]
as a direct sum of irreducible \( S_5 \)-modules.
(3) Find the decomposition of the induction product
\[ (S_{(2,1)} \otimes S_{(3,1)}) \uparrow_{S_7} \]
as a direct sum of irreducible \( S_7 \)-modules.

Problem 2: (202 Qual, S 2013) Decompose the restriction \( S_{(4,1)} \downarrow_{S_5} \) as a direct sum of irreducible \( S_3 \times S_2 \)-modules.

Problem 3: Prove that the symmetric group \( S_n \) has presentation given by generators
\[ s_1, s_2, \ldots, s_{n-1} \]
and relations
\[
\begin{aligned}
  s_i^2 &= e & 1 \leq i \leq n - 1 \\
  s_is_j &= s_js_i & |i - j| = 1 \\
  s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} & 1 \leq i \leq n - 2.
\end{aligned}
\]
(Hint: Show that the adjacent transpositions \((i,i+1)\) - which generate \( S_n \) - satisfy these relations. Then let \( G_n \) be the abstract group generated by these relations. Use induction to show that \(|G_n| \leq n!\).) This result (the ‘Coxeter presentation of \( S_n \)’) is a special case of a more general result (Matsumoto’s Theorem) in the field of Coxeter groups. It is very useful for constructing representations of \( S_n \) in practice. In general, presentations (of groups, algebras, ...) make constructing representations easier.

Problem 4: Prove the following dual Jacobi-Trudi identity. Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \) be a partition of \( n \) and let \( \lambda' \) be the conjugate of \( \lambda \). The Schur function \( s_{\lambda'} \) is given by the determinant
\[ s_{\lambda'} = \det(e_{\lambda_i-i+j})_{1 \leq i,j \leq k} \]
of elementary symmetric functions. (You could apply the involution \( \omega \) from the next problem, but it will probably be better to do this directly using Lindström-Gessel-Viennot theory. Make a strategic choice of planar network.)

Problem 5: The ring \( \Lambda \) of symmetric functions has algebraically independent generators \( e_1, e_2, \ldots \). Define a \( \mathbb{C} \)-algebra map \( \omega : \Lambda \to \Lambda \) by
\[ \omega(e_n) = h_n, \]
for all \( n \geq 1 \).

1. Prove that \( \omega \) is an involution (i.e., \( \omega^2 = \text{id} \)). (Hint: We have the alternating identity \( h_n - e_1 h_{n-1} + e_2 h_{n-2} - \cdots \pm e_n = 0 \) for all \( n \).)

2. Prove that \( \omega(s_\lambda) = s_{\lambda'} \) for all partitions \( \lambda \).

3. Prove that \( \omega \) is an isometry for the Hall inner product, i.e. for all \( f, g \in \Lambda \) we have
   \[ \langle f, g \rangle = \langle \omega(f), \omega(g) \rangle. \]

4. Give a formula for \( \omega(p_\lambda) \).

The map \( \omega \) is the symmetric function analog of tensoring by the sign representation.

**Problem 6:** Let \( \lambda \vdash n \) and let \( \lambda' \) be the conjugate of \( \lambda \). Prove that
\[ S^\lambda \cong S^\lambda' \otimes \text{sign}, \]
where sign is the sign representation of \( \mathfrak{S}_n \). (Hint: Murnaghan-Nakayama.) Now prove that the multiplicity of the sign representation of \( \mathfrak{S}_n \) in the Kronecker product
\[ S^\lambda \otimes S^{\lambda'} \]
is 1.

**Problem 7:** (Optional) For \( k \geq 0 \), define linear operators \( h_k^+ \) and \( e_k^+ \) on the ring \( \Lambda \) of symmetric functions by the formulas
\[ \langle h_k^+ f, g \rangle = \langle f, h_k g \rangle \]
and
\[ \langle e_k^+ f, g \rangle = \langle f, e_k g \rangle \]
for all \( f, g \in \Lambda \). That is, these are the adjoint operators (with respect to the Hall inner product) to multiplication by \( h_k \) and \( e_k \). This problem will give a representation theoretic interpretation of these operators.

1. Prove that the operators \( e_k^+ \) and \( h_k^+ \) are both homogeneous of degree \( -k \).
2. Consider the subgroup \( \mathfrak{S}_{n-k} \times \mathfrak{S}_k \) of \( \mathfrak{S}_n \). If \( \gamma \in \mathbb{C} [\mathfrak{S}_k] \) is any element in the group algebra of \( \mathfrak{S}_k \) and if \( V \) is any \( \mathfrak{S}_n \)-module, explain why (and how)
\[ \gamma V = \{ \gamma.v : v \in V \} \]
may be regarded as a \( \mathfrak{S}_{n-k} \)-module.
3. Define elements \( \eta_k, \varepsilon_k \in \mathbb{C} [\mathfrak{S}_k] \) by
   \[ \eta_k = \frac{1}{k!} \sum_{\pi \in \mathfrak{S}_k} \pi, \]
   \[ \varepsilon_k = \frac{1}{k!} \sum_{\pi \in \mathfrak{S}_k} \text{sign}(\pi) \cdot \pi. \]
   Prove that these operators are idempotents inside \( \mathbb{C} [\mathfrak{S}_k] \).
(4) Let $V$ be any $\mathfrak{S}_n$-module. Prove that

$$\text{ch}(\eta_k V) = h_k^\perp \text{ch}(V)$$

and that

$$\text{ch}(\varepsilon_k V) = e_k^\perp \text{ch}(V).$$

(Use Frobenius reciprocity.)

(5) Let $f, g \in \Lambda$ be any symmetric functions with equal constant terms. Prove that the following three statements are equivalent \(^1\)

(a) $f = g,$
(b) $h_k^\perp f = h_k^\perp g$ for all $k \geq 1,$
(c) $e_k^\perp f = e_k^\perp g$ for all $k \geq 1.$

Problem 8: (Optional) Let $G$ be a finite group and let $S$ and $T$ be finite sets, each of which carry an action of the group $G$. Let $\mathbb{C}[S]$ and $\mathbb{C}[T]$ be the corresponding $G$-modules. Consider the following two statements.

(1) There exists a bijection $\varphi : S \rightarrow T$ such that $\varphi(g.s) = g.\varphi(s)$ for all $s \in S.$
(2) There exists a $\mathbb{C}$-linear isomorphism $\psi : \mathbb{C}[S] \rightarrow \mathbb{C}[T]$ which commutes with the action of $G.$

Prove that (1) $\Rightarrow$ (2) but that the converse is in general false. \(^2\)

Problem 9: (Optional) Define a function $f$ on $3 \times 3$ matrices $A = (a_{ij})$ by

$$f(A) = a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31}.$$

Prove that, for any $3 \times 3$ square sub-matrix $A$ of the infinite matrix $H = (h_{j-i})_{i,j \geq 0},$ the symmetric function $f(A)$ expands into the Schur basis with positive coefficients. \(^3\)

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\(^1\)This result, combined with the last part, was an important component in an inductive step of my most recent project.

\(^2\)This was the subject of an awkward interchange with an anonymous journal referee. The statement (1) is not provable with character theory in general – the analog here is the far more difficult to apply ‘Frobenius table of marks’.

\(^3\)The matrix function $f$ is the Kazhdan-Lusztig immanant $\text{Imm}_{213}$ corresponding to the permutation $213 \in \mathfrak{S}_3.$ In general, there is a KL immanant $\text{Imm}_w$ - defined on $n \times n$ matrices - corresponding to any permutation $w \in \mathfrak{S}_n.$ When $w = e,$ we have $\text{Imm}_w = \det$ is the determinant function. You have just proven that the function $\text{Imm}_{213}$ is Schur positive. Haiman prove that $\text{Imm}_w$ is Schur positive for all permutations $w.$