§1.1 Review of Group Theory

* $S_n = \{ \text{all bijections } \sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \}$

"Symmetric group"

**Notation for permutations**

\[ \sigma \in S_6 \]

\[ \begin{array}{c}
\sigma(1) = 3 \\
\sigma(2) = 5 \\
\sigma(3) = 6 \\
\sigma(4) = 4 \\
\sigma(5) = 2 \\
\sigma(6) = 1
\end{array} \]

\[ \sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 4 & 2 & 1 \end{bmatrix} \]

\[ \sigma = 356421 \]  \[ \text{"one-line"} \]

"functional" \[ \sigma(1) = 3 \\
\sigma(2) = 5 \\
\sigma(3) = 6 \\
\sigma(4) = 4 \\
\sigma(5) = 2 \\
\sigma(6) = 1 \]

\[ \sigma = (1 \ 3 \ 6)(2 \ 5)(4) \]  \[ \text{"cycle"} \]

**Convention**

\[ (1 \ 2)(2 \ 3) = (1 \ 2 \ 3) \cdot (132)(125) \]

* If $G$ is a group \& $g, h \in G$, then $g, h$ are
conjugate $\iff \exists x \in G \text{ s.t. } g = xhx^{-1}$.

\[ \sigma = (1 \ 3 \ 6)(2 \ 5)(4) \]  \[ \text{(cycle)} \]

\[ \pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 6 & 1 & 3 & 4 \end{bmatrix} \]  \[ \text{(two-line)} \]

\[ \pi \sigma \pi^{-1} = (2 \ 6 \ 4)(5 \ 3)(1) \]

$\Rightarrow$ The cycle type of $\sigma \in S_n$ is a list of cycles of $\sigma$ written in decreasing order.

\[ \lambda = (\lambda_1, \lambda_2, \ldots) \]  \[ \text{cycle type} \]

\[ \text{eg} \quad \sigma = (1 \ 4)(2 \ 6 \ 5)(3 \ 7)(8) \to (3, 2, 2, 1) \]
**Fact** \( \sigma, \pi \in S_n \) are conjugate \( \iff \) \( \sigma, \pi \) have same cycle type.

* If \( G \) is a group \( \& \ g \in G \), the conjugacy class of \( g \) is \( K_g := \{ xgx^{-1}: x \in G \} \), and the centralizer is \( Z_g := \{ x \in G: xg = gx \} \).

\( \ast \) We have a 1-to-1 correspondence:

\[
\frac{G}{Z_g} \leftrightarrow K_g
\]

* A partition of \( n \geq 0 \) is a weakly decreasing list \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_e) \) of positive integers such that \( \lambda_1 + \cdots + \lambda_e = n \).

Write \( \lambda \vdash n \) or \( |\lambda| = n \).

**Ex** (4,3,2,1) \( \vdash 9 \) "Ferrers diagram" \[ \begin{array}{ccccccc} & & & & & & x \\
& & & & & x & \\
& & & & x & & \\
x & & & x & & & \\
& x & & x & & x & \\
& & x & & x & & \\
& & & x & & x & \\
& & & & x & & x \\
& & & & & x & x \\
& & & & & & x x \end{array} \]

**Def** If \( \lambda \vdash n \), let \( K_\lambda = \{ \sigma \in S_n: \sigma \text{ has cycle type } \lambda \} \).

\( \Box \) Given \( \lambda \vdash n \), \( |K_\lambda| = ? \)

**Fact** Let \( A = (a_1, a_2, \ldots, a_n) \) with \( \lambda \vdash n \) \& let \( m_i(\lambda) = \text{mult. of } i \text{ in } \lambda \).

Then \( |K_\lambda| = \frac{n!}{m_1(\lambda)! \cdot m_2(\lambda)! \cdots m_e(\lambda)!} = \varphi_\lambda. \)
eq \quad \sigma \in S_9 \text{ of cycle type } = \frac{9!}{4 \cdot 2 \cdot 2 \cdot 1 \cdot 1}.

Why? Take \( \lambda = 4331111 + 13 \)

\((**\,**)\, (**\,**)\, (**\,**)\, (\cdot)\, (\cdot)\, (\cdot)\)

- 13! ways to replace *'s w/ 1,2,3,...,13
- \(4!\cdot3\cdot13\) ways to rotate within cycles
- \(1!\cdot2!\cdot3!\) ways to permute cycles wholesale.

Def If \( \lambda \vdash n \), set \( Z_\lambda : = \frac{1}{n!} m_1(\lambda) ! m_2(\lambda) ! \ldots m_n(\lambda) ! \).

Then \( |K_\lambda| = n! / Z_\lambda \).

\[I\] If \( \sigma \in S_n \) has cycle type \( \lambda \) then \( |Z_\lambda| = Z_\lambda \).

* A two-cycle \( \tau = (i, j) \in S_n \) is a \underline{transposition}.

Fact If \( T = \{ (i, j) : 1 \leq i < j \leq n \} \) then \( S_n = \langle T \rangle \).

§ 1.2 Matrix reps of groups

\( k \)-field \( \text{GL}_n(k) = \{ \text{all invertible } n \times n \text{ matrices } / k \} \).

Def Let \( G \) be a group & let \( k \) be a field. A \underline{matrix representation of } \( G \) over \( k \) is a group homomorphism

\[ X : G \rightarrow \text{GL}_d(k), \quad \text{d is the degree of } X \quad (\deg X = d) \].
Ex 1  \[ G = \text{any group} \quad X : G \rightarrow GL_n(k) \cong k^* \]
\[ g \mapsto 1 \]
\[ \text{is the trivial representation (deg = 1)} \]

2  \[ G = S_n \quad X : S_n \rightarrow GL_n(k) \]
\[ \sigma \mapsto P_\sigma \quad "\text{permutation matrix}" \]
\[ \sigma = 231 \Rightarrow P_\sigma = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad (P_\sigma)_{ij} = \begin{cases} 1 & \sigma(j) = i \\ 0 & \text{else} \end{cases} \]

**Fact**  \[ X \text{ is a representation of } S_n \quad (\deg X = n) \]
\[ (P_\sigma \pi = P_\sigma \cdot P_\pi \text{ for all } \sigma, \pi \in S_n) \quad "\text{defining representation}" \]

3  \[ G = C_n = \langle g \mid g^n = e \rangle \quad k = \mathbb{C} \quad \zeta = \exp(2\pi i/n) \]
\[ \text{cyclic gp of order } n \]
\[ X_d : G \rightarrow GL_1(\mathbb{C}) \cong \mathbb{C}^* \]
\[ g \mapsto (\zeta^d) \quad d = 0, 1, \ldots, n-1 \]

4  \[ \text{The sign of } \pi \in S_n \text{ is } (-1)^k \text{ where} \]
\[ \pi = t_1 t_2 \cdots t_k \text{ for transpositions } t_i \in S_n. \]
\[ \text{The sign repn of } S_n \text{ is} \]
\[ X_{\text{sgn}} : S_n \rightarrow GL_1(k) = k^* \]
\[ \pi \mapsto \text{sgn } \pi. \]
\[ \Gamma \quad \text{sgn}(\pi \sigma) = (\text{sgn } \pi) \cdot (\text{sgn } \sigma) \quad \text{for all } \pi, \sigma \in S_n \]
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Last Time  Partitions  \( \lambda = (\lambda_1, \ldots, \lambda_k) \quad \epsilon \mathbb{Z}_{\geq 0} \)

\( \lambda \vdash n \iff \lambda_1 + \cdots + \lambda_k = n \)

\[ K_\lambda = \{ \pi \in S_n : \pi \text{ has cycle type } \lambda \} \]

\[ |K_\lambda| = \frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_k!} \]

\[ m_i = \# \text{ of } i \text{'s in } \lambda \]

\( \mathbb{Z}_\lambda = 1, m_2, m_3, \ldots, m_n \)

§ 1.2 Matrix Representations

Let \( G \) be a group. A \underline{matrix representation} of \( G \) over \( k \)

is a group homomorphism \( X : G \rightarrow GL_d(k) \) (for some \( d \)).

The degree of \( X \) is \( \text{deg } X = d \).

Example 1: \( G = S_n \)

\( X : S_n \rightarrow GL_d(k) \) where

\[ (p_\sigma) = \begin{cases} 1 & \sigma(i) = i \\ 0 & \text{otherwise} \end{cases} \]

\[ p_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \] "permutation matrix"

\( p_\sigma p_\pi = p_\sigma p_\pi \) for all \( \sigma, \pi \in S_n \).

\( X \) is the defining rep'n of \( S_n \).  \( \text{deg } X = n \).

Example 2:

\( G = S_n \)

\( X : S_n \rightarrow GL_1(k) \equiv k^\times ( = k - \{0\}) \)

\[ \sigma \mapsto \text{sgn } \sigma \]

\[ \text{sgn}(\sigma \pi) = \text{sgn } \sigma \cdot \text{sgn } (\pi) \]

(The sign/alternating rep'n of \( S_n \))  \( \text{deg } X = 1 \).
\[ (3) \quad G = \text{any group} \quad X^\text{triv} : G \rightarrow \text{GL}_1(k) \quad g \mapsto (i). \]

\[ X : \mathbb{Z} \rightarrow \text{GL}_2(k) \quad \mathbb{Z} \mapsto \left( \begin{array}{cc} 1 & \eta \\ 0 & 1 \end{array} \right). \]

\[ \left( \begin{array}{cc} 1 & \eta \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & \eta \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} 1 & \eta + \eta' \\ 0 & 1 \end{array} \right) \quad \text{deg} X = 2. \]

**Ex.** Classify all degree 1 \( \mathbb{C} \)-matrix reps of \( C_n = \langle g \mid g^n = e \rangle \).
A hom. \( X : C_n \rightarrow \text{GL}_1(\mathbb{C}) \cong \mathbb{C}^* \) is determined by \( X(g) \in \mathbb{C}^* \).

Also, \( X(g^n) = X(g)^n \)

so \( \zeta = e^{2\pi i/n}, X(g) \in \{1, \zeta, \zeta^2, \ldots, \zeta^{n-1}\} \).

\( X(e) = 1 \)

\( \Rightarrow \) For \( j = 0, 1, \ldots, n-1 \) have \( X : C_n \rightarrow \text{GL}_1(\mathbb{C}) \)

\[ g \mapsto (\zeta^j). \]

\[ \text{1.3} \quad \text{G-modules} \quad k = \text{field}. \]

**Def.** Let \( G \) be a group. A **G-module over** \( k \) is a \( \text{hom} \)-k-vector space \( V \) with a map \( G \times V \rightarrow V \) s.t.

\[ (1) \quad e \cdot v = v \quad \text{for all} \ v \in V \]

\[ (2) \quad g \cdot (\alpha v + \beta w) = \alpha (g \cdot v) + \beta (g \cdot w) \quad \text{for all} \ g \in G, \ \alpha, \beta \in k, \]

\[ (3) \quad (g \cdot h) \cdot v = g \cdot (h \cdot v) \quad \text{for all} \ g, h \in G \text{ and } v \in V. \]
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Lecture 3

1/11/19

$G$-group $K$-field $\rightarrow K$-v.s.

Matrix repn of $G$ $\rightarrow$ $G$-module $V$

$X : G \rightarrow GL_d(K)$ $\rightarrow$ $G \times V \rightarrow V$ $\rightarrow V \oplus W$

* If $X : G \rightarrow GL_d(K)$ $Y : G \rightarrow GL_e(K)$ are matrix repns of $G$, the direct sum is

$X \oplus Y : G \rightarrow GL_{d+e}(K)$

$X$ $g \mapsto X(g) \oplus Y(g) = \begin{bmatrix} X(g) & 0 \\ 0 & Y(g) \end{bmatrix}$

Ex: Let $G$ be a group, $H \subseteq G$ a subgroup.

$G \backslash H = \{ gH : g \in G \}$ (left cosets) $G \times G \backslash H \rightarrow G \backslash H$

$\Rightarrow K[G \backslash H]$ is a permutation $G$-module

$H = \{1\}$ $\Rightarrow$ regular rep'n of $G$

$H = G$ $\Rightarrow$ trivial rep'n of $G$.

Def 1.4 Submodules

Def: Let $V$ be a $G$-module. A $K$-linear subspace $W \subseteq V$ is a submodule if $g \cdot w \in W$ for all $g \in G$ and $w \in W$. 
\[ E \quad G = S_n \quad V = K^n \quad \sigma : (d_1, \ldots, d_n) \rightarrow (d_{\sigma(1)}, \ldots, d_{\sigma(n)}) \]

\[ W = K - \text{span} \{(i)\} \]

\[ U = \{(d_1, \ldots, d_n) \in K^n : d_1 + \cdots + d_n = 0\} \]

\[ \text{Rank} \quad 0, V \leq V \text{ are trivial submodules} \]

\[ \text{Def} \quad \text{A } G \text{-module } V \neq 0 \text{ is irreducible if the only submodules of } V \text{ are } 0, V. \]

- So, if \( n > 1 \), perm. rep. of \( S_n \) not irreducible.

- \( \mathbb{C} \otimes \mathbb{R}^2 \) "defining" is irreducible if for all \( n \geq 3 \).

\[ \text{Rank} \quad \text{If } V \text{ is a fin-dim } G \text{-module, WS } W \leq V, \text{ can find a basis } w_1, \ldots, w_m \text{ for } W \& \text{ extend to a basis } B = w_1, \ldots, w_m, v_1, \ldots, v_n \text{ of } V. \]

For all \( g \in G \),

\[ \begin{bmatrix} V \rightarrow V \\ v \rightarrow g.v \end{bmatrix} \text{ looks like } \begin{bmatrix} w_1 - w_m & v_1 - v_n \\ w_m & v_n \end{bmatrix} \]

\[ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \]
Def Let $V, W$ be $G$-modules. A linear map

$q : V \to W$ is a $G$-module homomorphism

if $q(g \cdot v) = g \cdot q(v)$ for all $g \in G, v \in V$.

$q$ is a $G$-module isomorphism if $q$ is invertible

(\& then $q^{-1} : W \to V$ is a $G$-homom.).

$V, W$ are isomorphic ($V \cong W$) if \exists a $G$-module isom. $V \to W$.

Ex Let $S_{n-1} \leq S_n$, so $V = K[S_n/S_{n-1}]$ is an $S_n$-module.

Let $W = K^n = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $a_i \in K$ permutation module.

Define $q : V \to W$

- $\sigma S_{n-1} \to e_{\sigma(i)}$

Check $q$ is an $S_n$-mod. isomorphism.

Rule If $q : V \to W$ is a $G$-mod. homom. then

$\text{Ker } q = \{v \in V : q(v) = 0\}$ is a submod of $V$

$\text{Im } q = \{q(v) : v \in V\}$ is a submod of $W$. 
Recall \( V = K \)-v.s. & \( W \subseteq V \) subspace

\[
\begin{align*}
V/W &= \{ u+W : u \in V \} \text{ quotient space}.
\end{align*}
\]

If \( W \subseteq V \) \( G \)-mod then \( V/W \) is a \( G \)-mod via \( G \)-submod.

\[
\begin{align*}
g \cdot (u+W) &= (g \cdot u) + W.
\end{align*}
\]

First issue. Thus if \( \Phi : V \to W \) is a \( G \)-mod. homom.

then \( \text{Im} \Phi \subseteq V/\ker \Phi \).

Q. What does this look like for matrices?

\[
\begin{align*}
X : G &\to GL_n(K) \quad \deg X = n \quad \text{matrix repns.} \\
Y : G &\to GL_m(K) \quad \deg Y = m
\end{align*}
\]

A \( G \)-homom. from \( X \) to \( Y \) is an \( m \times n \) matrix

\[
\begin{align*}
& \text{from } K^n \text{ to } K^m \quad A \in \text{Mat}_{m \times n}(K) \text{ st } \\
& G \times G \to Y
\end{align*}
\]

A \( X(g) = Y(g) A \) for all \( g \in G \).
LAST TIME

Submodules $W \subseteq V$ are $G$-module homomorphisms $\Theta : V \to W$.

Example $G = S_n$, $V = K^n$, $W = K[S_n/S_{n-1}]$ (defining repn) $S_{n-1} = \{ \pi \in S_n : \pi(n) = n \}$.

Claim $V \cong S_n \otimes W$, a $k$-linear map.

Proof Let $e_i = (\delta_{ij})$ for $1 \leq i \leq n$. Define $\Theta : W \to V$ by $\Theta : \pi S_{n-1} \mapsto e_\pi(\Theta)$. Then $\Theta$ is an $S_n$-homomorphism:

$$\sigma \cdot (\Theta(\pi S_{n-1})) = \sigma \cdot e_\pi(\Theta) = e_{\sigma \cdot \pi}(\Theta)$$

$\Rightarrow$

$$\Theta(\sigma \pi S_{n-1}) = e_{\sigma \cdot \pi}(\Theta) \checkmark$$

Also $\Theta$ is invertible. \(\blacksquare\)

Fact If $\Theta : V \to W$ is a $G$-module homomorphism then

$\text{Ker } \Theta := \{ \nu \in V : \Theta(\nu) = 0 \}$ is a submodule of $V$.

$\text{Im } \Theta := \{ \Theta(\nu) : \nu \in V \}$ is $W$.

* If $W \subseteq V$ is a $G$-submodule, then

$V/W = \{ \nu + W : \nu \in V \}$ is a $G$-module via

$g \cdot (\nu + W) = (g\nu) + W$.

First Isom Theorem: If $\phi : V \to W$ is a $G$-mod. homomorphism,

$\text{Im } \phi \cong V/\text{Ker } \phi$.  

What are $G$-homomorphisms in terms of matrices?

Let $X : G \to GL_n(K)$, $Y : G \to GL_m(K)$ be $G$-homomorphisms from $X$ to $Y$.

\[ \begin{pmatrix} K^n \to K^m \\ G \text{ via } X \circ G \text{ via } Y \end{pmatrix} \]

is an $m \times n$ matrix $T \in \text{Mat}_{m \times n}(K)$ such that

\[ TX(g) = Y(g)T \quad \text{for all } g \in G. \]

Recall a $G$-module $V \neq 0$ is irreducible if for any submodule $W \leq V$, $W = V$ or $W = 0$.

Def Let $V$ be a $G$-module and $W, U \leq V$ be $G$-submodules.

Write $V = W \oplus U$ if $W \cap U = 0$, $W + U = V$.

$V$ is indecomposable if $V = W \oplus U$ for $G$-submodules $(V \neq 0)$.

$U, W \to U = 0$ or $W = 0$.

* Irreducible $\Rightarrow$ Indecomposable.

\[ \Delta \text{ Not every indecomposable module is irreducible!} \]

$G = \mathbb{Z}$, $+$

$G = \mathbb{Z}/m\mathbb{Z}$, $\text{char } K = p$, $p \mid m$

$X : \mathbb{Z} \to GL_2(K)$

$Y : \mathbb{Z}/m\mathbb{Z} \to GL_2(K)$

$m \to \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$

not irreducible but indecomposable!
Def A G-module $V$ is **completely reducible** if

$$V = W_1 \oplus \cdots \oplus W_t$$

for irreducible submodules $W_1, \ldots, W_t \subseteq V$.

Mœschler's Thm Let $V$ be a fin-dim G-module / $K$. Assume that $|G| < \infty$ and either $\text{char } K = 0$ or $\text{char } K = p$ & $p \nmid |G|$. (\iff $\frac{1}{|G|}$ exists in $K$). Then $V$ is completely reducible.

If $|G| < \infty$, \ $\text{char } K = 0$ or \ $\text{char } K \nmid |G|$, then

irreducible $\iff$ indecomposable.

If Let $W \subseteq V$ be a $G$-submodule. We exhibit a $G$-subm. $U$ st $V = U \oplus W$. Then we are done by induction on $\dim V$.

Let $\pi : V \to V$ be a **linear projection** onto $W$ (so $\pi(w) = w$, $\pi(u) \not\in W$). Define a new $f_{\pi}$

$$\pi' : V \to V$$

$$u \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot u).$$

Then

- $\text{Im } \pi' \subseteq W$, $W$ a submod $\Rightarrow \text{Im } \pi' \subseteq W$
- If $w \in W$, $\pi'(w) = \frac{1}{|G|} \sum_{g \in G} g \cdot w = w$.
- $\pi'$ is **linear & a G-homom**:

If $g \in G, v \in V$

$$\pi'(h \cdot v) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot h \cdot v) = \frac{1}{|G|} \sum_{g \in G} h \cdot g \cdot \pi(g^{-1} \cdot u)$$

$$= h \cdot \pi'(v).$$
So \( V = \frac{\text{Im } \pi'}{\text{Ker } \pi'} \oplus \text{Ker } \pi' \)
\( W \) a \( G \)-submodule.

**Rmk.** In general, if \( V \) is a finitely gen'd \( G \)-module,
\[ F \text{ } G \text{-submodules} \]
\[ 0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_t = W \] s.t.
\[ U_i/U_{i-1} \text{ is irreducible for all } i. \] "Composition Series"

Also, the composition factors \( U_i/U_{i-1} \) are unique up to rearrangement/ isomorphism. "Jordan-Hölder"
Last Time

Maschke's Thm Assume $|G| < \infty$ & $V$ is a fin-dim'l $G$-module over $K$.
Assume $|G| \neq 0$ in $K$. Then $V = W_1 \oplus \cdots \oplus W_t$ for some irreducible submodules $W_1, \ldots, W_t \subseteq V$.

$\text{Rmk}$ $W_1, \ldots, W_t$ determined uniquely up to isom. / permutation.

Let $V, W$ be $G$-modules & $\Phi : V \rightarrow W$ be a $G$-module homomorphism. What does $\Phi$ look like?

Baby Schur's Lemma
1. If $V$ is irreducible, $\Phi = 0$ or $\Phi$ is injective.
2. If $W$ is irreducible, $\Phi = 0$ or $\Phi$ is surjective.
3. If $V$ and $W$ are irreducible, $\Phi = 0$ or $\Phi$ is an isomorphism.

$\text{Ker} \Phi \subseteq V$ and $\text{Im} \Phi \subseteq W$ are submodules.

True Schur's Lemma Let $V$ be an irreducible finite-dim'l $G$-module over $K$. Assume $K = K$ (algebraically closed).

Let $\Phi : V \rightarrow V$ be a $G$-mod. homom. $\exists a \in K$ st.

$\Phi(u) = au \quad \forall u \in V. \quad \text{[\Phi = a \cdot Id_V]}$

Pf Since $\dim V < \infty$ & $K = K$, the map $\Phi : V \rightarrow V$ has an eigenvector: $\exists u_0 \in V - \{0\}$ and $a \in K$ st $\Phi(u_0) = au_0$. (Jordan Form)

Now the map $\Phi - a \cdot Id_V : V \rightarrow V$ is a $G$-module homomorphism. $\Phi - a \cdot Id_V : u_0 \rightarrow 0$. By Baby Schur, since $V$ is irreducible, $\Phi - a \cdot Id_V \equiv 0$ so $\Phi = a \cdot Id_V$. 
$G = C_3 = \langle g \mid g^3 = e \rangle$

**Remark**

$\theta \in \mathbb{R}$

$V = \mathbb{R}^2$

$V$ is irreducible as a $C_3$-rep, but for any $\theta \in \mathbb{R}$,

$\phi_\theta : V \rightarrow V$ is a $G$-mod. homomorphism.

$\phi_\theta$ rotated by $\theta$

**Definition**

Let $V, W$ be $G$-modules over $K$

$\text{Hom}_G(V, W) = \{ \phi : V \rightarrow W \text{ a } G\text{-homomorphism} \}$

$\text{End}_G(V) = \{ \phi : V \rightarrow V \text{ a } G\text{-homomorphism} \}$

$\text{End}_G(V)$ forms a K-vector space

Also $\text{Hom}_G(X, Y)$, where $\text{End}_G(X)$ for matrix reps $X, Y$ of $G$.

**Schur's Lemma**

If $K = \mathbb{R}$ & $X$ is irreducible of degree $d$,

$\text{End}_G(X) = \{ \alpha \text{ Id} : \alpha \in \mathbb{R} \}$

$d \times d$

What about $\text{End}_G(X \oplus Y)$?

$X : G \rightarrow \text{GL}_d(\mathbb{R})$ noniso. irreduct.

$Y : G \rightarrow \text{GL}_e(\mathbb{R})$

Let $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ in $\text{Mat}_{d \times e}(\mathbb{R})$

$T \in \text{End}_G(X \oplus Y) \iff \begin{bmatrix} X(g) & 0 \\ 0 & Y(g) \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} X(g) & 0 \\ 0 & Y(g) \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \forall g \in G$

$\iff X(g)T_{11} = T_{11} X(g), \ X(g)T_{12} = T_{12} Y(g) \forall g \in G$

$\iff Y(g)T_{21} = T_{21} X(g), \ Y(g)T_{22} = T_{22} Y(g)$

$\iff T_{12} = 0, \ T_{21} = 0, \ T_{11} = \alpha \text{ Id}, \ T_{22} = \beta \text{ Id} \quad (\alpha, \beta \in \mathbb{R})$
So \( \text{End}_G(X \oplus Y) = \left\{ \begin{pmatrix} \alpha I_d & 0 \\ 0 & \beta I_e \end{pmatrix} : \alpha, \beta \in K \right\} \)

nonisom. irreds

\[ \Rightarrow \dim \text{End}_G(X \oplus Y) = 2 \left\{ \alpha I_d \oplus \beta I_e : \alpha, \beta \in K \right\} \cong K^2. \]

Q:
What about \( \text{End}_G(X \oplus X) \) for \( X \) irreducible?

Ans. \( X : G \rightarrow \text{GL}_d(K) \), \( \deg X = d \), \( T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \text{Mat}_{d+d}(K) \).

\[ T_{11} x(g) = x(g) T_{11} , \quad T_{12} x(g) = x(g) T_{12} \quad \forall g \in G \]

\[ T_{21} x(g) = x(g) T_{21} , \quad T_{22} x(g) = x(g) T_{22} \]

Schur \( \Leftrightarrow \exists \alpha, \beta, \gamma, \delta \in K \) s.t.

\[ T_{11} = \alpha I_d , \quad T_{12} = \beta I_d \]

\[ T_{21} = \gamma I_d , \quad T_{22} = \delta I_d \]

isom. irreducible

\[ \Rightarrow \left\{ \begin{pmatrix} \alpha I_d & \beta I_d \\ \gamma I_d & \delta I_d \end{pmatrix} : \alpha, \beta, \gamma, \delta \in K \right\} \]

\[ \text{tensor product} \]

If \( A = (a_{ij}) \)

\[ A \otimes B = (a_{ij}B a_{ij}B \ldots) \in \otimes^r \text{Mat}_d(K) \Rightarrow \dim \text{End}_G(X \oplus X) = 4. \]

In general: Assume \( X : G \rightarrow \text{GL}_d(K) \) is a matrix rep'n with \( |G| < \infty \), \( |G| \neq 0 \) in \( K \), and \( K = \overline{K} \) (Maschke + Schur).

Write \( X \cong \bigoplus_{i=1} \bigoplus_{m_i} X_{i}^{\oplus m_i} \bigoplus \cdots \bigoplus \bigoplus_{r} X_{r}^{\oplus m_r} \). Then ...

\[ \text{End}_G(X) = \left\{ \bigoplus (A_i \otimes I_{d_i}) : A_i \in \text{Mat}_{m_i}(K) \right\} \bigoplus \text{Mat}_{m_r}(K). \]

So \( \dim = m_1^2 + \cdots + m_r^2. \)
Similarly,

$$\dim_K \text{Hom}_G(X_i, X) = m_i = \text{multiplicity of } X_i \text{ in } X$$

* From now on, $G = \text{finite}$, $V = \text{finite-dimensional}$, $K = \overline{K}$ (Maschke + Schur)

**CHARACTERS**

Q. Given a $G$-module $V$, is $V$ irreducible?

Q. Given a $G$-module $V$ & an irreducible $G$-module $W_j$, multiplicity of $W$ in $V = W_1 \oplus \ldots \oplus W_t \oplus \text{irreducibles}$?

**Def** Let $X: G \to \text{GL}_d(K)$ be a matrix rep'n of $G$.

The **character** of $X$ is $\chi_X: G \to K$

$$g \mapsto \text{trace } (X(g))$$

If $V$ is a $G$-module, $\chi_V: G \to K$

$$g \mapsto \text{trace } (V \to V)$$

**Fact**

1. If $\deg X = d$ then $\chi_X(e) = \text{trace } (I_d) = d$.

2. We have $\chi_{X \otimes Y}(g) = \chi_X(g) + \chi_Y(g)$.

3. For any $g, h \in G$,

$$\chi_X(hgh^{-1}) = \text{tr } (X(h)X(g)X(h^{-1})) = \text{tr } (X'(h)) = \chi_X(g)$$

$\Rightarrow \chi_X$ invariant on conjugacy classes.
Last Time

Schur's Lemma \( V = \text{irreducible fin-dim G-module}/K \). \( 
\Phi: V \to V \)

G-mod. endom. \( \text{If } 
\exists \ \lambda \in K \text{ s.t. } \Phi = \lambda \cdot \text{Id}_V. \)

* Assume \( G = \text{finite} \) \( V = \text{fin-dim} \) \( K = \mathbb{C} \).

\[ X: G \to \text{GL}_d(\mathbb{C}) \quad \forall \ \mu \in G \]

matrix rep' \( \quad \text{End}_G(X) = ? \)

- If \( X: G \to \text{GL}_d(\mathbb{C}) \), \( Y: G \to \text{GL}_e(\mathbb{C}) \) are irreducible,

\( \text{End}_G(X \otimes Y) = \{ \lambda \text{Id}_d \otimes \mu \text{Id}_e : \lambda, \mu \in \mathbb{C} \} \cong \mathbb{C}^{d \times e} \quad (2 \text{-dim} ) \)

\( \text{End}_G(X \otimes X) = \{ A \otimes \text{Id} : A \in \text{Mat}_d(\mathbb{C}) \} \cong \text{Mat}_d(\mathbb{C}). \quad (4 \text{-dim} ) \)

Fact: If \( X = X^{(1)} \oplus \ldots \oplus X^{(r)} \) for

non-iso. irreducible matrix rep's

\( X^{(1)}, \ldots, X^{(r)} \) of degrees \( d_1, \ldots, d_r \) & \( m_1, \ldots, m_r > 0 \)

then

\( \text{End}_G(X) = \{ (A_1 \otimes \text{Id}) \oplus \ldots \oplus (A_r \otimes \text{Id}) : A_i \in \text{Mat}_{m_i}(\mathbb{C}) \} \)

\( \cong \text{Mat}_{m_1}(\mathbb{C}) \oplus \ldots \oplus \text{Mat}_{m_r}(\mathbb{C}) \leftarrow \text{dim} = m_1^2 + \ldots + m_r^2. \)

Similar \( \quad \text{dim} \left( \text{Hom}_G(X^{(i)}, X) \right) = m_i = \text{dim} \left( \text{Hom}_G(X, X^{(i)}) \right). \)

\( \Rightarrow \text{Irreducible Multiplicities well defined!} \).
Def: Let $X: G \to GL_d(\mathbb{C})$ be a matrix repion of $G$

The character of $X$ is $\chi_X: G \to \mathbb{C}$
$$g \mapsto \text{trace } X(g)$$

Let $V$ be a $G$-module. The character is
$$\chi_V: G \to \mathbb{C}$$
$$g \mapsto \text{trace } \left[ V \to V \right]_{\omega \to g.\omega}$$

Ex 1 $G = S_3$, $V = \mathbb{C}^2$, $\chi_V: S_3 \to \mathbb{C}$

$\chi_V(e) = 2$
$\chi_V(12) = 0 = \chi_V(23) = \chi_V(13)$

$$(123) \mapsto \begin{bmatrix} \cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} \\ \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \xrightarrow{\text{tr}} -1$$

So $\chi_V: (123) \mapsto -1$
$\chi_V: (321) \mapsto -1$.

2 $X = \text{finite } G\text{-set} \Rightarrow G \circlearrowleft \mathbb{C}[x]$

$$\chi_{\mathbb{C}[x]}(g) = \frac{1}{|G|} \sum_{x \in X} \chi_{\mathbb{C}[x]}(g) = \frac{|X^g|}{|G|}$$
**FACT** 1. \( \chi_{V}(e) = \dim V \), \( \chi_{x}(e) = \deg x \).

2. \( \chi_{x}(hgh^{-1}) = \text{tr}(X(hgh^{-1})) = \text{tr}(X(h)x(g)X(h)^{-1}) = \text{tr}(X(g)) = \chi_{x} \).

For \( h, g \in G \),

\( \chi_{V}(hgh^{-1}) = \chi_{V}(g) \).

3. If \( X \cong Y \in T \) s.t. \( X(g) = T Y(g) T^{-1} \), \( \forall g \in G \).

So,

\[ \chi_{x}(g) = \text{tr} X(g) = \text{tr} T Y(g) T^{-1} = \text{tr} Y(g) = \chi_{Y}(g). \]

(Converse also)

(\text{true} ! ! ! ! !)

**Def** A function \( \chi: G \to \mathbb{C} \) is a class function if

\( \chi(xgx^{-1}) = \chi(g) \) for all \( x, g \in G \).

\( \mathcal{R}(G) = \{ \text{all class functions } G \to \mathbb{C} \} \)

\( \Rightarrow \) a \( \mathbb{C} \)-vector space

\( \dim \mathcal{R}(G) = \# \text{ of conjugacy classes of } G. \)

**Ex**

\[ \begin{array}{c|ccc|c}
\text{Irred. class} & 1 & 3 & 2 & \# \text{ of elts} \\
\hline
\chi^{\text{triv}} & 1 & 1 & 1 & \# \text{ of elts} \\
\chi^{\text{sgn}} & 1 & -1 & 1 & \# \text{ of conj. classes} \\
\chi^{2d} & 2 & 0 & -1 & \\
\end{array} \]

**Def** Let \( \langle , \rangle \) be the inner product on \( \mathcal{R}(G) \) given by

\( \langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \psi(g). \)
FACT If \( V \) is any \( G \)-module \( \mathbb{C} \) and \( g \in G \),
the operator \( V \rightarrow V \) is unitary.

\[
\exists \text{ a } \mathcal{B} \text{-basis of } V \text{ & } A_g = [V \rightarrow V] \text{ the } A_g^{-1} = A_g^*.
\]

Why? Let \( \langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C} \) be any inner product on \( V \).
Define \( \langle \cdot, \cdot \rangle' : V \times V \rightarrow \mathbb{C} \)
by \( \langle u_1, u_2 \rangle' = \frac{1}{|G|} \sum_{g \in G} \langle g.u_1, g.u_2 \rangle \) for \( u_1, u_2 \in V \).
Then \( \langle \cdot, \cdot \rangle' : V \times V \rightarrow \mathbb{C} \) is also an inner product & \( \langle g.u_1, g.u_2 \rangle' = \langle u_1, u_2 \rangle' \) for all \( u_1, u_2 \in V \).
"G-invariant"

Now let \( \mathcal{B} \) be an orthonormal basis for \( V \) w.r.t. \( \langle \cdot, \cdot \rangle \).
Check if \( b_i, b_j \in \mathcal{B} \) and \( g \in G \) then
\[
\alpha (A_g)_{ij} = \text{ coeff. of } b_j \text{ in } g.b_i' = \langle g.b_i, b_j \rangle' = \langle b_i, g^{-1}.b_j \rangle'
\]
\[
= \langle g^{-1}.b_j, b_i \rangle'
= \text{ complex conj. of coeff. of } b_i \text{ in } g^{-1}.b_j
= (A_{g^{-1}})_{ji}
\]

\[ \Rightarrow \forall V \text{ is a } \mathcal{G} \text{-mod w/ char. } \chi_V : G \rightarrow \mathbb{C}, \chi_V(g') = \chi_V(g) \text{ for all } g, g' \in G. \]
Last Time \( X : G \to \text{GL}_n(\mathbb{C}) \) matrix rep.

Character \( \chi = \chi_X : G \to \mathbb{C} \quad \chi(g) = \text{trace } X(g) \)

\[ \chi(g^{-1}) = \overline{\chi(g)} \quad \chi(gag^{-1}) = \chi(g) \quad \forall \ g, x \in G. \]

\( R(G) = \{ \text{all reps } G \to \mathbb{C} \text{ constant on conj. classes} \} \) “class func”

\[ \langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \psi(g) \]

Character Orthogonality of the First Kind

Let \( A, B : G \to \text{GL}_n(\mathbb{C}) \), \( B : G \to \text{GL}_m(\mathbb{C}) \) be irreducible

Matrix reps of \( G \). Let \( \chi = \text{character of } A \), \( \psi = \text{character of } B \).

Then \( \langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } A \cong_B B \\ 0 & \text{if } A \not\cong_B B \end{cases} \)

Proof For any \( g \in G \), write \( A(g) = (a_{ij}(g))_{1 \leq i, j \leq n} \)

\( B(g) = (b_{ij}(g))_{1 \leq i, j \leq m} \)

Suppose \( X = (x_{ij})_{1 \leq i, j \leq n} \) is an \( n \times m \) matrix of variables.

Define a new matrix \( Y \) by \( Y = \frac{1}{|G|} \sum_{g \in G} A(g) \times B(g^{-1}) \)

Then for \( x \in G \),

\[ Y \cdot B(x) = \frac{1}{|G|} \sum_{g \in G} A(g) \times B(g^{-1}x) = \frac{1}{|G|} \sum_{g \in G} A(xg) \times B(g^{-1}) = A(x) \cdot Y, \]

so \( Y \) is a homomorphism from \( B \) to \( A \).
Suppose \( A \not\equiv G \). Then \( Y \) is the zero matrix by Schur's Lemma. So if \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \)

\[
0 = (i,j)\text{- entry of } Y = (i,j)\text{- entry of } \frac{1}{|G|} \sum_{g \in G} A(g) \times B(g^{-1})
\]

\[
\langle a_{ir}, b_{sj} \rangle := \frac{1}{|G|} \sum_{g \in G} a_{ir}(g) \cdot x_{rs} \cdot b_{sj}(g^{-1})
\]

\[
\sum_{g \in G} \langle a_{ir}(g), b_{sj}(g^{-1}) \rangle = \sum_{r=1}^{n} \sum_{s=1}^{m} \langle a_{ir}, b_{sj} \rangle \cdot x_{rs}.
\]

This means that \( \langle a_{ir}, b_{sj} \rangle = 0 \) for all \( 1 \leq i, r \leq n, 1 \leq s, j \leq m \).

So

\[
\langle \chi, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \text{tr} \ A(g) \times B(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} \left( \sum_{i=1}^{n} a_{ii}(g) \right) \left( \sum_{j=1}^{m} b_{jj}(g^{-1}) \right)
\]

\[
= \frac{n^2}{|G|} \sum_{i=1}^{n} \sum_{j=1}^{m} \langle a_{ii}, b_{jj} \rangle = 0.
\]

Now assume \( A \equiv B \). WLOG \( A = B \). Schur's Lemma implies that \( \exists \ \lambda \in \mathbb{C} \) s.t. \( Y = \lambda \mathbf{I}_n \). We still have \( \langle a_{ir}, b_{sj} \rangle = 0 \) whenever \( i \neq j \). Furthermore, \( 1 \leq i \leq n \)

\[
n \cdot \lambda = \text{tr} \ Y = \text{tr} \ \frac{1}{|G|} \sum_{g \in G} A(g) \times A(g^{-1}) = \text{tr} \ X, \text{ so } \ y_{ii} = \lambda = \frac{1}{n} \cdot \text{tr} \ X.
\]

This means

\[
y_{ii} = \sum_{r=1}^{n} \sum_{s=1}^{n} \langle a_{ir}, a_{si} \rangle \cdot x_{rs} = \frac{1}{n} \cdot (x_{ii} + \cdots + x_{nn}).
\]

This means \( \langle a_{ir}, a_{si} \rangle = 0 \) unless \( i = r = s \) and \( \langle a_{ij}, a_{ij} \rangle = \frac{1}{n} \).

So

\[
\langle \chi, \varphi \rangle = \sum_{i,j} \langle a_{ii}, a_{jj} \rangle = \sum_{i=1}^{n} \langle a_{ii}, a_{ii} \rangle = \sum_{i=1}^{n} \frac{1}{n} = 1.
\]
Fact: \( \mathbb{C}[G] \cong \bigoplus_{\text{irred } V} V \oplus \dim V \)

\( V \) an irred
\( G \)-mod

(take dimensions
\( |G| = \sum_r (\dim V)^2 \). "Magic formula"

\( V \) an irred
\( G \)-module

Goal: \( \{ \chi: G \to \mathbb{C} : \chi \text{ an irreducible character} \} \) is an orthonormal basis for \( \mathbb{C}(G) \). (We know it's orthogonal so linearly independent...)

Recall: \( A = \text{algebra} \)

\( Z_A = \text{center of } A = \{ z \in A : az = za \text{ for all } a \in A \} \).

\( \text{Ex 1} \) \( A = \text{Mat}_{n \times n}(\mathbb{C}) \Rightarrow Z_A = \{ \lambda I_n : \lambda \in \mathbb{C} \} \)

\( \dim Z_A = 1 \)

2. \( Z_{A \oplus B} = Z_A \oplus Z_B \)

3. \( A = \text{Mat}_m(\mathbb{C}) \times \cdots \times \text{Mat}_r(\mathbb{C}) \Rightarrow \dim Z_A = m \cdot \prod_{i=1}^r m_i, \quad (m_1, \ldots, m_r > 0) \)

So, if \( V = G \)-module \& \( V = W_1 \oplus \cdots \oplus W_r \), \( m_1, \ldots, m_r > 0 \)

for nrs \( W_1, \ldots, W_r \) then

\( \text{dim } \text{End}_G(V) = \text{Mat}_m(\mathbb{C}) \oplus \cdots \oplus \text{Mat}_r(\mathbb{C}) \) (Schaar)

\( Z_{\text{End}_G V} = \bigoplus_{\text{irred } V} V \oplus \dim V = r. \)
Let $V$ be a $G$-module and write

$$V = W_1 \oplus \cdots \oplus W_r$$

where $W_1, \ldots, W_r$ are pairwise nonisom. $G$-modules.

1. We have the equality of characters

$$\chi_V = m_1 \chi_{W_1} + \cdots + m_r \chi_{W_r}.$$

2. $m_i = \langle \chi_V, \chi_{W_i} \rangle$

3. $\langle \chi_V, \chi_V \rangle = m_1^2 + \cdots + m_r^2$

4. $V$ is irreducible $\iff \langle \chi_V, \chi_V \rangle = 1$.

**Example** Let $\mathbb{C}[G] = \text{regular repn. of } G$ w/ character $\chi_{\text{reg}} : G \to \mathbb{C}$. Let $\chi$ be any irreducible character of $G$.

$$\chi_{\text{reg}} : G \to \mathbb{C}, \quad \chi_{\text{reg}}(g) = 1.$$ 

We have $\chi_{\text{reg}}(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$.

Let $V$ be any irreducible $G$-mod, $\chi_V : G \to \mathbb{C}$ character.

**multiplicity of $V$ in $\mathbb{C}[G]$**

$$\langle \chi_V, \chi_{\text{reg}} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \chi_{\text{reg}}(g)$$

$$= \frac{1}{|G|} \cdot |G| \cdot \chi_V(e) = \dim V.$$

$\Rightarrow$ Every irreducible $G$-module $V$ appears in $\mathbb{C}[G]$ w/ multiplicity $= \dim V$. 
LAST TIME

G < finite gp & V, W = irreducible G-mods

χ_v, χ_w : G → C characters

* \[ \langle χ_v, χ_w \rangle = \begin{cases} 1 & \text{if } V \cong G W \\ 0 & \text{if } V \not\cong G W \end{cases} \]

Cor: If V, W are any G-mods then \( V \cong G W \iff χ_v = χ_w \).

Cor: \( C[G] \cong \bigoplus_{G \text{-irreps } V} V^\oplus \dim V \) (so \( |G| = \sum (\dim V)^2 \).

Goal: \# of irreducible chars of G = \# of conjugacy classes of G.

Def: Let A be a C-algebra. The center of A is

\( Z_A = \{ z \in A : za = az \text{ for all } a \in A \} \) (a C-subalgebra).

Ex: 1. \( A = \text{Mat}_n(C) \Rightarrow Z_A = \{ \lambda I_n : \lambda \in \mathbb{C} \} \Rightarrow \dim Z_A = 1 \).

2. \( Z_{A \oplus B} = Z_A \oplus Z_B \).

If V is a G-module s.t. \( V \cong \bigoplus G W_1^\oplus \cdots \oplus W_r^\oplus \)

v with \( m_i > 0 \) and \( W_1, \ldots, W_r \) nonisom. irreducible G-modules

then… \( \text{End}_G V \cong \text{Mat}_{m_1}(C) \oplus \cdots \oplus \text{Mat}_{m_r}(C) \)

\( \Rightarrow \dim Z_{\text{End}_G V} = \frac{1 + \cdots + 1}{r} = r \).
Let $G$ be a finite group.

\[
\text{# of noniso. } G\text{-mods} = \text{# of conjugacy classes } W = \sum_{K \leq G} \text{dim } W.
\]

We know that $C[G] \cong \bigoplus_{\text{all } G\text{-irreps} \ W} W$, so

\[
\text{# of noniso } G\text{-mods} = \text{dim } \mathcal{Z}_{\text{End}_G C[G]}.
\]

We have a map

\[
\Psi : C[G] \rightarrow \text{End}_G (C[G])
\]

\[
\text{a} \mapsto (b \mapsto ba)
\]

Then $\Psi$ is a $G$-linear isomorphism.

(Inverse: \( \Phi : \text{End}_G C[G] \rightarrow C[G] \))

\[
\Phi : b \mapsto \phi(b)
\]

Also $\Psi(1_{\alpha_1}) = \Psi(1_{\alpha_2})$ (algebra anti-linear).

So $\Psi$ induces an algebra isomorphism

\[
\mathcal{Z}_{C[G]} \cong \mathcal{Z}_{\text{End}_G C[G]},
\]

so

\[
\text{# of noniso } G\text{-mods} = \text{dim } \mathcal{Z}_{C[G]}.
\]

Let \[
\sum_{g \in G} \alpha_g g \in C[G] \quad (\alpha_g \in \mathbb{C}).
\]

Then

\[
\sum_{g \in G} \alpha_g g \in \mathcal{Z}_{C[G]} \iff \forall z \in G \quad \sum_{g \in G} \alpha_g g = \sum_{g \in G} \alpha_g (z g z^{-1})
\]

\[
= \alpha_g z g z^{-1} \quad \forall \ g, z \in G.
\]
So \[ \sum_{g \in g} \chi \longrightarrow \delta : G \rightarrow \mathbb{C} \]
\[ g \mapsto \chi_g \text{ is a class func!} \]

Thus \[ \dim \mathbb{Z}[CG] = \dim \mathbb{R}(G) = \# \text{ of conjugacy classes of } G. \]

* The **character table** of a group \( G \) is ...

\[
\begin{array}{c|ccc}
\text{irred. chars} & \chi & \cdots & \chi_{K} \\
\hline
\text{conj. classes} & K & \vdots & \\
S_{3} & 1 & 3 & 2 \\
G = \mathbb{R}_{3} & (1,2,3) & (1,2,3) & (1,2,3) \\
\chi^{1} & 1 & 1 & 1 \\
\chi^{2} & 1 & -1 & 1 \\
\chi^{3} & 2 & 0 & -1 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\text{irred. chars} & \chi & \cdots & \chi_{K} \\
\hline
\text{conj. classes} & K & \vdots & \\
C_{4} & 1 & g^{2} & g^{3} \\
G = C_{4} = \langle g | g^{4} = e \rangle & 1 & i & -i \\
\chi^{0} & 1 & 1 & 1 \\
\chi^{1} & 1 & i & -i \\
\chi^{2} & 1 & -1 & -1 \\
\chi^{3} & 1 & -i & i \\
\end{array}
\]

**THM**: \( \{ \chi : G \rightarrow \mathbb{C} : \chi \text{ an irreducible character} \} \) is an orthogonal basis for \( \mathbb{R}(G) \).
Character Orthogonality of 2nd kind

Let $G$ be a finite gp & let $C, K \leq G$ be conjugacy classes. Then

$$\sum_{\chi : G \to \mathbb{C}} \overline{\chi_C} \chi_K = \begin{cases} 0 & C \neq K \\ |G|/|K| & C = K \end{cases}$$

A square matrix with orthogonal rows has orthonormal columns.

**Ex.** Character table of $D_4$?

$D_4 = \langle s, r \mid s^2 = r^4 = e, srs = r^{-1} \rangle$

Conjugacy classes:

- $\{e\}$
- $\{x^2\}$
- $\{r, x^3\}$
- $\{s, sx^2\}$
- $\{sr, sx^3\}$

<table>
<thead>
<tr>
<th>$r$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^1$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^2$</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi^3$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^4$</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi^5$</td>
<td>0</td>
</tr>
</tbody>
</table>

"defining" $\chi^5 = 2 -2 0 0 0$
Last time \[ G \text{ # of } G \text{-irreps} = \text{ # of conj. classes} \text{ non-isom.} \]

\[ K \subseteq G \]

\[ \Rightarrow \{ \chi: G \to \mathbb{C} : \chi \text{ an irred. char} \} \text{ is an ON basis for } \mathbb{R}(G). \]

**Character Table**

\[ \begin{array}{c|cccc}
\chi & K & \text{cong. class} \\
\hline
\chi_K & 1 & \chi & -\chi_K \\
\text{irred. char.} & K & & & \\
\end{array} \]

**Ex:** \( C_4, D_4, S_4 \)

**Char orth. of 2nd kind**

If \( K, C \subseteq G \) are conj. classes then

\[ \sum_{\chi: G \to \mathbb{C}} \chi_K \chi_C = \begin{cases} 0 & K \neq C \\ |G|/|K| & K = C \end{cases} \]

Recall if \( A, B \) are matrices, \( A \oplus B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \)

**Def** Let \( X: G \to \text{GL}_n(\mathbb{C}) \), \( Y: H \to \text{GL}_m(\mathbb{C}) \) be matrix reps of \( G, H \). Then

\[ X \oplus Y: G \times H \to \text{GL}_{n \times m}(\mathbb{C}) \]

\[ (g, h) \mapsto X(g) \oplus Y(h) \]

**Check** \( X \oplus Y \) is a (well-defined!) repn of \( G \times H \).

\[ (A_1 \oplus A_2) \cdot (B_1 \oplus B_2) = (A_1B_1) \oplus (A_2B_2) \]

**Characters** \( \chi_X: G \to \mathbb{C} \), \( \chi_Y: H \to \mathbb{C} \)

\[ \chi_X \otimes \chi_Y: G \times H \to \mathbb{C} \]

Characters. Then \( \chi_X \otimes \chi_Y (g, h) = \chi_X (g) \cdot \chi_Y (h) \forall g \in G, h \in H \).
What is $\otimes$ for modules? $(V, W \rightarrow V \otimes W)$.

Def Let $V, W$ be $C$-vector spaces. Then $V \times W = \{(v, w) : v \in V, w \in W\}$.

$C[V \times W] = C$-v.s. w/ basis $\in V \times W$.

$V \otimes W = \text{quotient space of } C[V \times W] \text{ consisting of smallest subspace of } C[V \times W] \text{ containing...}$

- $(v + v', w) - (v, w) - (v', w)$
- $(v, w + w') - (v, w) - (v, w')$
- $(\alpha v, w) - \alpha (v, w)$
- $(v, \alpha w) - \alpha (v, w)$

for all $v \in V, w \in W$.

$V \otimes W = \text{image of } (v, w)$ in $V \otimes W$.

Ex $V = C[x], W = C^2$.

\[ V \otimes W \ldots \]

\[ (2x^2 + 7) \otimes (-1) = (2x^2 \otimes (-1)) + (7 \otimes (-1)) \]

\[ = 2(x^2 \otimes (-1)) + (7 \otimes (-1)) + (7 \otimes (1)) \]

\[ = [x^2 \otimes (-1)] + 25 \cdot [1 \otimes (0)] + 28 \cdot [1 \otimes (1)] \]

\[ - 2[x^2 \otimes (0)] + 8 \cdot [x^2 \otimes (1)] \]
**Fact** Suppose \( B \) is a basis for \( V \). Then \( B \otimes \{ v \otimes w : v \in B, w \in W \} \) is a basis for \( V \otimes W \).

**Universal Property of \( \otimes \)** Let \( U, V, W \) be \( \mathbb{C} \)-vector spaces.

Let \( \varphi : V \times W \rightarrow U \) be a function such that \( \varphi \) is **bilinear**:

\[
\begin{align*}
\varphi (u + u', w) &= \varphi (u, w) + \varphi (u', w) \\
\varphi (u, w + w') &= \varphi (u, w) + \varphi (u, w') \\
\varphi (\alpha u, w) &= \varphi (u, \alpha w)
\end{align*}
\]

**Claim** \( V \otimes W \cong C[x,y] \).

**Proof**

Let \( \varphi : C[x] \times C[y] \rightarrow C[x,y] \) defined by \( \varphi(f(x), g(y)) = f(x)g(y) \).

Then \( \varphi \) is **bilinear** so we have a (well-defined) \( \mathbb{C} \)-linear map

\[
\tilde{\varphi} : C[x] \otimes C[y] \rightarrow C[x,y]
\]

\[f(x) \otimes g(y) \mapsto f(x)g(y)\].
Define \( \Psi: \mathbb{C}[x,y] \rightarrow \mathbb{C}[x] \otimes \mathbb{C}[y] \) linear by
\[ x^i y^j \mapsto x^i \otimes y^j. \]

Check that \( \Phi \circ \Psi = \text{Id}_{\mathbb{C}[x,y]} \) and \( \Psi \circ \Phi = \text{Id}_{\mathbb{C}[x,y]} \).

RMK If \( u \in V, \omega \in W \) then \( u \otimes \omega \in V \otimes W \) is a "simple tensor".

Not all els of \( V \otimes W \) are simple tensors!

\[ V, W = \mathbb{C}^2 = \mathbb{C}(e_1, e_2) \quad (e_1 \otimes e_2) + (e_2 \otimes e_1) \in \mathbb{C}^2 \otimes \mathbb{C}^2 \]
\[ \not\in \text{not a simple tensor}. \]
\[ \neq (ae_1 + be_2) \otimes (ce_1 + de_2) \]
for any \( a, b, c, d \in \mathbb{C} \).

But \( V \otimes W \) is spanned by simple tensors.

Connection to Matrix Tensor? \( V_1, W_1, V_2, W_2 = \mathbb{C}\)-v.s.

\( \varphi_1: V_1 \rightarrow W_1 \quad \varphi_2: V_2 \rightarrow W_2 \) = linear maps

\( \Rightarrow \) Get a linear map \( \varphi_1 \otimes \varphi_2: V_1 \otimes V_2 \rightarrow W_1 \otimes W_2 \)

\( \alpha_1 \otimes \alpha_2 \mapsto \varphi_1(\alpha_1) \otimes \varphi_2(\alpha_2). \)

\( B_1 = \) ordered basis for \( V_1 \)
\( B_2 = \) ordered basis for \( V_2 \)

\( \mathbf{e}_1 = \) basis for \( W_1 \)
\( \mathbf{e}_2 = \) basis for \( W_2 \)

\( [\varphi_1]_{B_1}^e = \) matrix for \( \varphi_1 \)
\( [\varphi_2]_{B_2}^e = \) matrix for \( \varphi_2 \)

\( [\varphi_1 \otimes \varphi_2]_{B_1 \otimes B_2}^{e_1 \otimes e_2} = [\varphi_1]_{B_1}^e \otimes [\varphi_2]_{B_2}^e \)

lexicographical order...
Last Time \( V, W = \mathbb{C}\)-vector spaces \( \Rightarrow V \otimes W \)

Universal Property of \( V \otimes W \)

Let \( U \) be a \( \mathbb{C}\)-v.s. & let \( \varphi: V \times W \rightarrow U \) be bilinear.

\[ \exists! \ C\text{-linear map } \Phi: V \otimes W \rightarrow U \text{ s.t. } \]
\[ \Phi(v \otimes w) = \varphi(v, w). \]

Claim \( C[x] \otimes C[y] \cong C[x, y] \) as \( \mathbb{C}\)-vector spaces

If define \( \varphi: C[x] \times C[y] \rightarrow C[x, y] \)
\[ (f(x), g(y)) \mapsto f(x)g(y). \]

Then \( \varphi \) is bilinear, so induces a \( \mathbb{C}\)-linear map
\[ \Phi: C[x] \otimes C[y] \rightarrow C[x, y] \]
\[ f(x) \otimes g(y) \mapsto f(x)g(y). \]

Now define a \( \mathbb{C}\)-linear map \( \Phi^*: C[x, y] \rightarrow C[x] \otimes C[y] \)
\[ x^i y^j \mapsto x^i \otimes y^j. \]

Check that \( \Phi \circ \Phi^* = \text{id}_{C[x, y]} \) & \( \Phi^* \circ \Phi = \text{id}_{C[x] \otimes C[y]} \)

* If \( \varphi_1: V_1 \rightarrow W_1 \), \( \varphi_2: V_2 \rightarrow W_2 \) are \( \mathbb{C}\)-linear, have a \( \mathbb{C}\)-linear map \( \varphi_1 \otimes \varphi_2: V_1 \otimes V_2 \rightarrow W_1 \otimes W_2 \).

Fact \( V_i = \) ordered basis of \( V_i \) \((i = 1, 2)\)
\[ e_i \] is a matrix tensor

then
\[ [\varphi_1 \otimes \varphi_2] e_i e_j = [\varphi_1] e_i [\varphi_2] e_j \]
\[ e_i e_j \] (use lexicographical order on \( e_i e_j, e_i e_j \))
Def. Let $V$ be a $G$-module & let $W$ be an $H$-module.

Then $V \otimes W$ is a $(G \times H)$-module via
\[
(g, h) \cdot (v \otimes w) := (g v) \otimes (h w)
\]
for $g \in G$, $h \in H$, $v \in V$, $w \in W$.

Restriction & Induction

Def. Let $H \leq G$ be a subgroup and let $X : G \to \text{Gl}_n$ be

a matrix rep'n of $G$. The restriction of $X$ to $H$ is $X : H \to \text{Gl}_n$,

the restriction of the homomorphism $X$.

\[
\text{Res}_H^G(X) = \begin{cases} V & \text{a } G\text{-mod} \rightarrow V \downarrow_H^G = V, \text{ as a } H\text{-mod.} \\
\end{cases}
\]

Q. Given a matrix rep'n $X : H \to \text{Gl}_n$, how to get

a matrix rep'n of $G$?

Def. Let $H \leq G$ be a subgroup and let $T = \{t_1, ..., t_r\}$

be a left transversal for $H$ in $G$.

Let $X : H \to \text{Gl}_n$ be a matrix rep'n of $H$.

The induction $X^G : G \to \text{Gl}_{nr}$ is given by

\[
\text{Ind}_H^G(X) = \begin{bmatrix}
X(t_1 g t_1) & \cdots & X(t_1 g t_r) \\
& & \\
& & \\
X(t_r g t_1) & \cdots & X(t_r g t_r)
\end{bmatrix}
\]

$X(t_i g t_j) = 0 \quad \forall \ t_i g t_j \notin H$. 

(Ind)$_H^G(X)$
\( X: G \rightarrow GL_1 \)

\[ \text{trivial rep'n!} \]

\[ X(t_i^{-1} g t_j) = \begin{cases} 
1 & \text{if } t_i^{-1} g t_j \in H \\
0 & \text{else}
\end{cases} \]

So \( X^G_H \) is the 
\text{coset rep'n } \frac{G}{H} 

\[ \text{triv}_H \uparrow^G \equiv C[\frac{G}{H}] \]

**THM**
Let \( H \leq G \) be a subgroup & let \( T = \{t_1, \ldots, t_r\} \) be a left transversal for \( H \) in \( G \). Let \( X: H \rightarrow GL_n \) be a matrix rep'n. The \text{induction } X^G \) is a well-defined rep'n of \( G \) & independent of \( T \) up to \( G \)-isom.

**LTP**
Let \( g \in G \)

For \( 1 \leq i \leq r \), \( t_i g t_j \in H \Leftrightarrow g t_j \in t_i H \).

For \( 1 \leq j \leq r \), \( t_i g t_j \in H \Leftrightarrow t_j^{-1} g^{-1} t_i \in H \)

\[ \Leftrightarrow g t_i \in t_j H \]

So just one blk of each blk row/col of \( (X(t_i^{-1} g t_j)) \) is \( 1 \) & that blk is \text{invertible}.

So \( X^G(g) \in GL_n \) for all \( g \in G \).
Given \( g, j \in G \), \( t_i^{-1} g t_j \in H \) & \( t_j^{-1} g' t_k \in H \) then \( t_i^{-1} g g' t_k \in H \) &

\[ X(t_i^{-1} g t_j) X(t_j^{-1} g' t_k) = X(t_i^{-1} g' t_k) \]

is a \text{homomorphism}.
Let $\varphi: H \to C$ be the character of $X$.

The character $\varphi^G: G \to C$ of $X^G \mu$

$\varphi^G(g) = \sum_{i=1}^{r} \varphi(t_i^{-1} g t_i)$, where $\varphi(t_i^{-1} g t_i) = 0$ unless $t_i^{-1} g t_i \in H$.

We have

$\varphi^G(g) = \sum_{i=1}^{r} \varphi(t_i^{-1} g t_i)$

$\varphi \in \mathcal{R}(G)$

$\varphi = \frac{1}{|H|} \sum_{h \in H} \sum_{i=1}^{r} \varphi(h t_i^{-1} g t_i h)$

$G = \prod_{i=1}^{r} t_i H$

$= \frac{1}{|H|} \sum_{z \in G} \varphi(z^{-1} g x)$, which is independent of $T$.

Def Let $H \leq G$ be a subgroup & let $\varphi: H \to C$ be a class function. The induction $\varphi^G: G \to C$

is $\varphi^G(g) = \frac{1}{|H|} \sum_{x \in G} \varphi(x^{-1} g x)$

where $\varphi(x^{-1} g x) = 0$ if $x^{-1} g x \notin H$.

* If $\varphi: G \to C$ is a class func, the restriction

is $\varphi|_H: H \to C$

$h \mapsto \varphi(h)$.

Frobenius Reciprocity Let $\varphi \in \mathcal{R}(H)$, $\psi \in \mathcal{R}(G)$. Then $\langle \varphi^G, \psi \rangle_G = \langle \varphi, \psi|_H \rangle_H$.

$\langle \varphi^G, \psi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \psi(g) = \frac{1}{|G|} \frac{1}{|H|} \sum_{g \in G} \varphi(x^{-1} g x) \overline{\psi(x g x)} = \frac{1}{|G|} \frac{1}{|H|} \sum_{x \in G} \varphi(x^{-1} g x) \overline{\psi(x g x)}$