Problem 1. First, let \( f \in k[x_1, \ldots, x_n] \) and consider \( f + I \in k[x_1, \ldots, x_n]/I \). Let \( r \) be the remainder of division of \( f \) by \( G \), then by the division algorithm \( \text{LM}(g_i) \nmid m \) for any \( i \) and any monomial \( m \) appearing in \( r \). Now, write \( r = \sum c_m m_\alpha \), where each \( m_\alpha \) satisfies \( \text{LM}(g_i) \nmid m_\alpha \) for all \( i \). Since \( G \subseteq I \), we have \( f + I = r + I = \sum c_m m_\alpha + I = \sum c_m (m_\alpha + I) \), hence \( \{ m + I : m \text{ monomial with } \text{LM}(g_i) \nmid m \text{ for all } i \} \) is a spanning set for \( k[x_1, \ldots, x_n]/I \).

To show that this set is a basis, we need linear independence. Suppose we have a linear combination
\[
\sum \alpha c_\alpha (m_\alpha + I) = 0
\]
for all \( \alpha \). Then \( \sum \alpha c_\alpha m_\alpha = 0 \), hence \( \sum \alpha c_\alpha m_\alpha \in I \). Since each \( m_\alpha \) is a monomial, it suffices to show that \( f = 0 \), since this will imply that all \( c_\alpha \) are 0. Let \( r \) be the remainder of \( f \) after division by \( G \). Since \( G \) is a Gröbner basis and \( f \in I \), we have \( r = 0 \). On the other hand, since no monomial of \( f \) is divisible by any \( \text{LM}(g_i) \) we have \( r = f \), hence \( f = r = 0 \), as desired.

It is essential for this proof that \( G \) is a Gröbner basis and not just any basis for \( I \). The proof that the monomials are a spanning set works even when \( G \) is just a basis, but the linear independence will usually fail. However, a lot of proofs failed to mention where it was essential that \( G \) is a Gröbner basis.

Problem 2.

1. A long computation shows that it is given by \( G = \{yx - x^2 + 2x, y^2 - y - 2x^2 + 4x, x^3 + x^2 - 6x\} \).
2. By the Elimination Theorem this is given by \( G \cap \mathbb{C}[x] = \{x^3 + x^2 - 6x\} \).
3. Since \( x^3 + x^2 - 6x \in I \), any solution \((x, y)\) should have \( x^3 + x^2 - 6x = 0 \), hence \( x \in \{0, 2, -3\} \).
4. By problem 1 this is given by all \( m + I \) with \( m \) a monomial not divisible by any of \( yx, y^2 \) and \( x^3 \), hence \( \{1 + I, x + I, x^2 + I, y + I\} \).

I know computing Gröbner basis isn’t the most exciting thing to do, but with a problem like this assigned, one should at least show some details. For part 4, note that \( \{1, x, x^2, y\} \) technically isn’t a basis for \( \mathbb{C}[x, y]/I \), but rather that it descends to a basis.

Problem 3.

1. Again a long computation shows that this is \( G = \{u - x - 2z, v - z, xz - y + 2z^2\} \).
2. By the closure theorem and the elimination theorem, \( V = V(I \cap k[x, y, z]) = V((G \cap k[x, y, z])) = V(xz - y + 2z^2) \).
3. Since we don’t know what field we’re working over, we can’t apply the Extension theorem. However, let \((x_0, y_0, z_0) \in V(xz - y + 2z^2) \). Set \( u = x_0 + 2z_0 \) and \( v = z_0 \). Then clearly \( x_0 = u - 2v \) and \( z_0 = v \) and furthermore, \( y_0 = x_0 z_0 + 2z_0^2 = z_0 (x_0 + 2z_0) = uv \), hence \((x_0, y_0, z_0) \in S \).

Since the problem didn’t specify the field we’re working, I didn’t subtract points for a correct application of the extension theorem. However, it’s good to take note of the above solution that works even in the case that the field is not algebraically closed.

Problem 4.

1. Note that
\[
h_d(x_1, \ldots, x_i) = \sum_{1 \leq i_1 \leq \ldots \leq i_d \leq i} x_{i_1} \cdots x_{i_d} = \sum_{1 \leq i_1 \leq \ldots \leq i_d = i} x_{i_1} \cdots x_{i_d} + \sum_{1 \leq i_1 \leq \ldots \leq i_d < i} x_{i_1} \cdots x_{i_d}
\]
\[
= x_i \sum_{1 \leq i_1 \leq \ldots \leq i_d = i-1} x_{i_1} \cdots x_{i_d} + \sum_{1 \leq i_1 \leq \ldots \leq i_d \leq i-1} x_{i_1} \cdots x_{i_d}
\]
\[
= x_i h_{d-1}(x_1, \ldots, x_{i-1}) + h_d(x_1, \ldots, x_{i-1}).
\]
(2) Set \( h_{d,i} = h_d(x_1, \ldots, x_i) \). In the above notation \( h_{d,i-1} = h_{d,i} - x_i h_{d-1,i} \). Therefore, if \( h_{d,i} \) and \( h_{d-1,i} \) belong to \( I_n \), we find that \( h_{d,i-1} \) belongs to \( i \). Now, by definition \( h_{1,n}, h_{2,n}, \ldots, h_{n,n} \) belong to \( I_n \). Applying the above observation we find that \( h_{2,n-1}, \ldots, h_{n,n-1} \in I_n \). Now, by induction, \( h_{j+1,n-j}, \ldots, h_{n,n-j} \) belong to \( I_n \) for any \( 0 \leq j \leq n-1 \). In particular, given \( 1 \leq i \leq n \), let \( j = i - 1 \), then we find that \( h_{i}(x_1, \ldots, x_{n-i+1}) = h_{i,n-i+1} = h_{j+1,n-j} \in I_n \), as desired.

(3) Let \( G = \{g_1, \ldots, g_k\} \) be a Gröbner basis for \( I_n \). For a monomial \( m \), we have \( \text{LM}(g_i) \nmid m \) if and only if \( m \notin \langle \text{LM}(I_n) \rangle \), since \( G \) is a Gröbner basis. Now, note that \( \langle \text{LM}(I_n) \rangle \) contains \( \{\text{LM}(h_1(x_1, \ldots, x_n)), \text{LM}(h_2(x_1, \ldots, x_{n-1})), \ldots, \text{LM}(h_n(x_1))\} = \{x_n, x_{n-1}^2, \ldots, x_1^2\} \). Consequently, if \( m \notin \langle \text{LM}(I_n) \rangle \), we necessarily have \( x_n \nmid m, \ldots, x_1^2 \nmid m \), i.e. \( m \) is a staircase monomial.

By problem 1 and the earlier observation a basis for \( R_n \) is given by \( \{m + I_n : m \text{ a monomial, } m \notin \langle \text{LM}(I_n) \rangle \} \subseteq \{m + I_n : m \text{ a staircase monomial} \} \), as desired.

The first two parts usually didn’t seem to cause too many problems. However, in the last part, some students tried to find a Gröbner basis for \( I_n \). However, since one only needs to show that the given monomials are a spanning set, and not a basis, the argument is actually a lot easier, and doesn’t need to find the entire Gröbner basis, but just show that the leading monomials of the elements from part (2) already impose enough conditions.