Math 202c   Lecture 15   5/11/18

LAST TIME — (char $k=0$) Suppose $G \subseteq GL_n(k)$ is a finite subgroup st

$$k[x_1,\dots,x_n]^G = k[f_1,\dots,f_r]$$

for algebraically indep. homog. $f_1,\dots,f_r \in k[\underline{x}_n]^G$

① $r=n$. ② $d_i \equiv \deg f_i$ are uniquely determined "invariant degrees" ③ $|G| = d_1 \cdot d_2 \cdots d_n$.

Hilbert's Thm Let $G \subseteq GL_n(k)$ be a finite group. The invariant ring $k[x_1,\dots,x_n]^G$ is a finitely generated k-algebra: $k[x_1,\dots,x_n]^G = k[\underbrace{f_1,\dots,f_r}_{\text{finite!}}]$ for homogeneous $f_1,\dots,f_r \in k[\underline{x}_n]^G$.

Ex $k[x,y]^{\pm\left(\begin{smallmatrix}1&\\&1\end{smallmatrix}\right)} = k[\underbrace{x^2, xy, y^2}_{\text{finite}}]$.

Recall Reynolds operator
$$R_G : k[\underline{x}_n] \longrightarrow k[\underline{x}_n]$$
$$f \longmapsto \frac{1}{|G|}\sum_{g\in G}(g\cdot f).$$

- $R_G(f) \in k[\underline{x}_n]^G$ always.
- $R_G(f) = f$ for $f \in k[\underline{x}_n]^G$
- R_G preserves degree
- If $f \in k[\underline{x}_n]^G$ & $h \in k[\underline{x}_n]$, $R_G(f\cdot h) = f\cdot R_G(h)$.

Pf of Hilbert Let
$$k[\underline{x}_n]^G_+ = \{f \in k[\underline{x}_n]^G : f(0,\dots,0)=0\}$$
$$= \bigoplus_{d\geq 1} k[\underline{x}_n]^G_d.$$
(a graded k-v.s.)

Let $I = \langle k[\underline{x}_n]^G_+ \rangle \subseteq k[\underline{x}_n]$. Then by <u>Hilbert's</u> <u>Basis</u> <u>Theorem</u>, $\exists$ finitely many <u>homogeneous</u> invariants $f_1, \ldots, f_r \in k[\underline{x}_n]^G_+$ s.t. $I = \langle f_1, \ldots, f_r \rangle$.

[<u>CLAIM</u> $\{f_1, \ldots, f_r\}$ generates $k[\underline{x}_n]^G$ as a k-algebra:

$$k[\underline{x}_n]^G = k[f_1, \ldots, f_r].$$]

We prove $(*)$:

$(*)$ Let $f \in k[\underline{x}_n]^G$ be <u>homogeneous</u> of degree $d \geq 0$.

Then $f \in k[f_1, \ldots, f_r]$.

We induct on d. If $d = 0$, then $f \in k$ and this is clear, so assume $d > 0$.

Let $d_i = \deg f_i$, ~~of~~ $1 \leq i \leq r$. Since $f \in I$, $\exists$ $h_1, \ldots, h_r \in k[\underline{x}_n]$ st

$$(**) \quad \overset{\deg d}{f} = h_1 \overset{d_1}{f_1} + \cdots + h_r \overset{d_r}{f_r}.$$

WLOG h_i is <u>homog</u> of degree $d - d_i$. Now apply R_G to both sides of $(**)$:

$$(\dagger) \quad \underset{\underset{f}{\parallel}}{R_G(f)} = \overset{\deg\, d-d_1}{R_G(h_1)} \cdot f_1 + \cdots + \overset{d-d_r}{R_G(h_r)} \cdot f_r$$

~~&~~ By induction (since $d_i > 0$), $R_G(h_i) \in \cancel{k[\underline{x}_n]^G}$ $k[f_1, \ldots, f_r]$, so

$f \in k[f_1, \ldots, f_r]$. //

Q How to make this effective?

Noether's Theorem Let $G \subseteq GL_n(k)$ be a finite matrix group. $k[\underline{x}_n]^G$ is generated as a k-algebra by

$$\left\{ R_G(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) \;:\; \alpha_1 + \cdots + \alpha_n \leq |G| \right\}$$

Ex $\zeta = \exp(2\pi i/3)$, $G = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} \zeta & \\ & \zeta^{-1} \end{pmatrix}, \begin{pmatrix} \zeta^{-1} & \\ & \zeta \end{pmatrix} \right\}$.

$G \circlearrowright \mathbb{C}[x,y]$ $\qquad \zeta \cdot \zeta^{-2} = \zeta^{-1}$

m	$R_G(m)$	m	$R_G(m)$
1	1	x^3	x^3
x	$\frac{1}{3}(1+\zeta+\zeta^2)x = 0$	xy^2	$\frac{1}{3}(xy^2 + \zeta xy^2 + \zeta^2 xy^2) = 0$
y	0	x^2y	0
x^2	0	y^3	y^3
xy	xy		
y^2	0		

$\Rightarrow \mathbb{C}[x,y] = \mathbb{C}[x^3, y^3, xy]$.

Pf of Noether Abbreviate (for $\alpha = (\alpha_1, -, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$)

$$x^\alpha \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_n} \;; \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n .$$

$k[\underline{x}_n]^G$ is gen'd by $\{ R_G(x^\beta) : |\beta| \geq 1 \}$. So if we fix $|\beta| = d \geq 1$, ETS $R_G(x^\beta)$ is a poly. in $\{ R_G(x^\alpha) : 1 \leq |\alpha| \leq |G| \}$.

Fix $d \geq 1$ & consider the identity

$$(*) \quad (y_1 + \cdots + y_n)^d = \sum_{|\beta| = d} a_\beta \, y^\beta, \quad a_\beta \in \mathbb{Z}_{>0}.$$

To exploit $(*)$, for any $A \in GL_n(k)$ let $A_i = i^{\text{th}}$ row of A,

$$A_i \cdot \underline{x} = A_i \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{j=1}^{n} a_{ij} \cdot x_j .$$

Also let $\quad A \cdot \underline{x}^\beta = (A_1 \underline{x})^{\beta_1} \cdots (A_n \cdot \underline{x})^{\beta_n}$.

Introduce variables $u_1, \ldots, u_n$ & evaluate $(*)$ at $y_i = (A_i \cdot x) u_i$:

$$(**) \quad \underbrace{(A_1 \cdot \underline{x}\, u_1 + \cdots + A_n \cdot \underline{x}\, u_n)}_{\equiv\, U_A}{}^d = \sum_{|\beta| = d} a_\beta \, (A \cdot \underline{x}^\beta) \cdot u^\beta .$$

Now sum $(**)$ over $A \in G$ & multiply by $\frac{1}{|G|}$:

$$(\dagger) \quad \frac{1}{|G|} \cdot \sum_{A \in G} U_A^d = \sum_{|\beta| = d} a_\beta \cdot R_G(\underline{x}^\beta)\, u^\beta \equiv S_d$$

The LHS of $(\dagger)$ is <u>symmetric</u> in the $|G|$ quantities $\{U_A : A \in G\}$.

So ~~if~~ $S_d = F(S_1, S_2, \ldots, S_{|G|})$ for some polynomial F.

~~Bsc~~ So if we take coeff. of u^β on both sides of $(\dagger)$ $(|\beta| = d)$, we get

$$\underset{\neq 0}{a_\beta} \cdot R_G(\underline{x}^\beta) = \left(\text{a polynomial in } R_G(\underline{x}^\alpha) \text{ for } 1 \leq |\alpha| \leq |G|\right).$$

∎

Math 202C   Lec. 16   5/14/18

Last Time $G \subseteq GL_n(k)$ finite matrix gp.

Hilbert's Thm $\exists$ finitely many homogeneous $f_1, \dots, f_r \in k[\underline{x}_n]^G_+$ s.t.

$$k[\underline{x}_n]^G_+ = k[f_1, \dots, f_r].$$

⌜Coming from Pf: If $f_1, \dots, f_r \in k[\underline{x}_n]^G_+$ are homogeneous & $\langle k[\underline{x}_n]^G_+ \rangle = \langle f_1, \dots, f_r \rangle$ then $k[\underline{x}_n]^G = k[f_1, \dots, f_r]$. ⌟

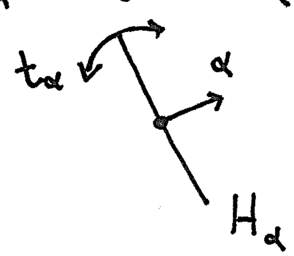
Noether's Thm   In fact, if $R_G(f) = \frac{1}{|G|} \sum_{g \in G} (g.f)$ then

$$k[\underline{x}_n]^G = k\left[R_G(\underline{x}^\alpha) \;:\; 0 \leq \alpha_1 + \dots + \alpha_n \leq |G| \right].$$

Reflection Gps V = Euclidean space ⌜fin-dim $\mathbb{R}$-vector space w/ symmetric, positive definite bilinear form $\langle \, , \rangle : V \times V \to \mathbb{R}$ ⌟

- A (linear) hyperplane $H \subseteq V$ is a codimension 1 subspace.

- Given $\alpha \in V - \{0\}$, $H_\alpha = \{ \beta \in V : \langle \alpha, \beta \rangle = 0 \}$.



⌜For $c \in \mathbb{R}^\times$, $H_\alpha = H_{c\cdot\alpha}$ ⌟

- For $\alpha \in V - \{0\}$, the reflection thru H_α is $t_\alpha : V \longrightarrow V$

$$t_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha. \quad ⌜ t_\alpha = t_{c\cdot\alpha} \text{ if } c \neq 0 ⌟$$

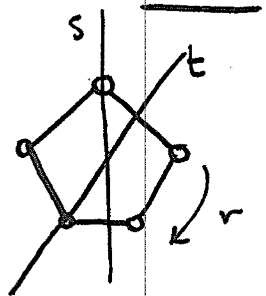
Fact $*$ $t_\alpha(\alpha) = -\alpha$; $t_\alpha(\beta) = \beta$ if $\beta \in H_\alpha$.

$*$ $t_\alpha \in O(V)$ (orthogonal)

$*$ $(t_\alpha)^2 = \mathrm{id}_V$. $*$ $t_\alpha \sim \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ in a suitable basis.

Def A finite subgp $W \subseteq GL(V)$ is a <u>reflection gp</u> if W is generated by reflections.

Ex ① $V = \mathbb{R}^2$: $W = \dfrac{\text{dihedral gp of}}{m\text{-gon symmetries}}$ $(m \geq 3)$



$(t \cdot s = r)$

$W = \langle s, r \mid s^2 = r^m = 1, \; srs = r^{-1} \rangle$

type $I_2(m)$; $= \langle s, t \mid s^2 = t^2 = 1, \; (st)^m = 1 \rangle$

m reflns

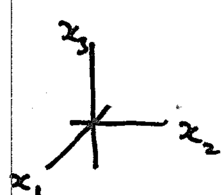
② $V = \{x_1 + \cdots + x_n = 0\} \subseteq \mathbb{R}^n$ $\qquad S_n \circlearrowright V$

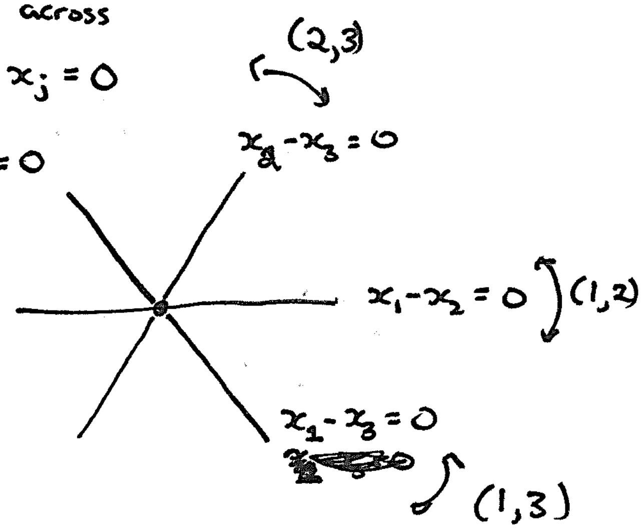
$\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

For $1 \leq i < j \leq n$

$(i, j) \in S_n \longleftrightarrow$ refl'n across $x_i - x_j = 0$

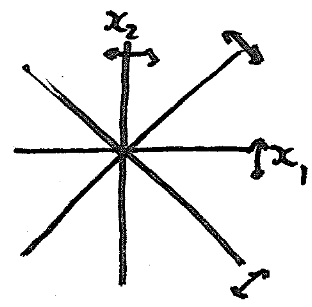
<u>$n = 3$</u>





type A_{n-1}; $\binom{n}{2}$ reflns

③ $V = \mathbb{R}^n$ $\qquad W = \{n \times n \text{ signed perm. matrices}\}$ $\qquad n = 2$



<u>reflns</u>

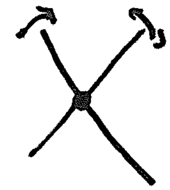
$x_i - x_j = 0$	$(1 \leq i < j \leq n)$	$i \mapsto j$, $j \mapsto i$
$x_i + x_j = 0$	$(1 \leq i < j \leq n)$	$i \mapsto -j$, $j \mapsto -i$
$x_i = 0$	$(1 \leq i \leq n)$	$i \mapsto -i$.

type B_n / C_n; $2 \cdot \binom{n}{2} + n = n^2$ reflns.

④ $W = \left\{ \begin{matrix} n\times n \text{ signed p.m.'s}, \\ \text{even \# of } -1\text{'s} \end{matrix} \right\}$ $\quad \begin{pmatrix} & 1 & & & \\ & & & & 1 \\ & & & -1 & \\ 1 & & & & \\ & & -1 & & \end{pmatrix}$

<u>reflns</u> $\quad x_i - x_j = 0 \quad 1 \le i < j \le n$

$x_i + x_j = 0 \quad 1 \le i < j \le n$

$n=2$: 

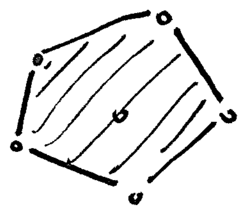
$\#$ of reflns $= 2 \cdot \binom{n}{2} = n \cdot (n-1)$. (type D_n)

<u>Q</u> Where do refln gps come from?

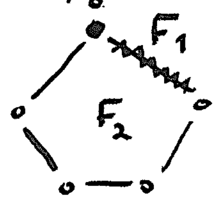
<u>A</u> ① Symmetry gps of <u>regular</u> <u>polytopes</u>.

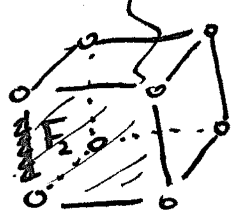
② Weyl gps.

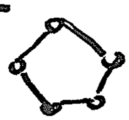
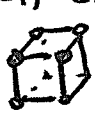
<u>Def</u> A <u>polytope</u> P is the convex hull of a finite point set in $\mathbb{R}^n$.



P is <u>regular</u> if its symmetry gp G acts <u>transitively</u> on <u>flags</u> of <u>faces</u>





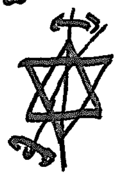
Reg. <u>Polytope P</u>	<u>Symmetry gp G</u>		
reg. m-gon 	$I_2(m)$ $(m \ge 3)$	regular dodecahedron $\subset \mathbb{R}^3$	H_3
 $\Delta^{n-1} = \{x_1 + \cdots + x_n = 1\}$ $(n-1)$-dim'l simplex	S_n (A_{n-1})	120-cell $\subset \mathbb{R}^4$	H_4
 /  n-cube / n-cross polytope	B_n / C_n	24-cell $\subset \mathbb{R}^4$	F_4

That's all! ~~Aside~~

(A2)

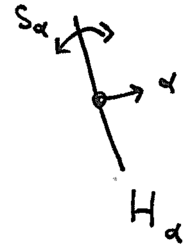
Simple complex compact Lie group G	Weyl gp W
$SU(n)$	S_n (A_{n-1})
$SO(2n+1) / Sp(2n)$	B_n / C_n
$SO(2n)$	D_n

$+ \quad G_2 = I_2(6), \; F_4, \; E_6, E_7, E_8$



LAST TIME – V = Euclidean space fin-dim $\mathbb{R}$-v.s. / inner prod $\langle -,- \rangle$

- $\alpha \in V - \{0\} \rightsquigarrow H_\alpha$ hyperplane $\rightsquigarrow s_\alpha$ refl'n.



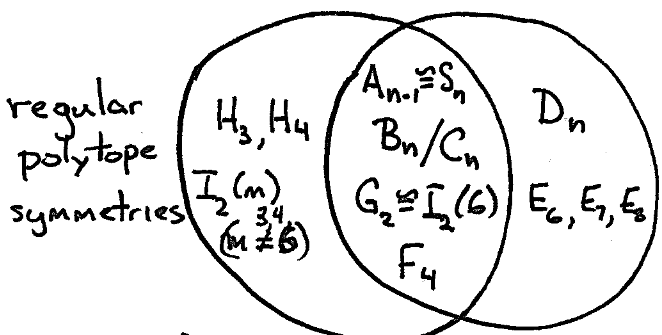
- A finite subgp $G \subseteq GL(V)$ is a reflection group if it is generated by reflns.

---

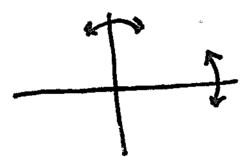
* Given a compact simple complex Lie group G, the "Weyl gp" W is a refln gp.

Lie Gp	$SU_n(\mathbb{C})$	$SO_{2n+1}(\mathbb{C})$	$Sp_{2n}(\mathbb{C})$	$SO_{2n}(\mathbb{C})$
Weyl Gp	S_n (A_{n-1})	B_n	C_n	D_n

$G_2 \cong I_2(6)$ + F_4 E_6 E_7 E_8



$\underline{\underline{\text{Ex}}}$ $\left\{ \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix} \right\}$ is a refl'n gp:



"$A_1 \times A_1$"

$\underline{\underline{\text{Def}}}$ Let $G \subseteq GL(V)$ be a refl'n group. If V can be written as an orthogonal direct sum $V = V_1 \oplus V_2$ s.t. every reflecting hyperplane H in G has $V_1 \subseteq H$ or $V_2 \subseteq H$, then G is reducible. [$\Rightarrow G \cong G_1 \times G_2$ for refln gps $G_i \subseteq GL(V_i)$.]

<u>FACT</u> The irreducible refl'n gps are <u>exactly</u>:

$$A_{n-1},\ B_n/C_n,\ \underset{(n\geq 4)}{D_n},\ H_3, H_4, F_4, E_6, E_7, E_8, \underset{(m\geq 3)}{I_2(m)}.$$

<u>Q</u> How are refl'n gps related to <u>invariant</u> <u>theory</u>?

<u>Theorem</u> Let $G \subseteq GL_n(\mathbb{R})$ be a finite subgroup. TFAE

① G is a reflection group.

② $\exists^n$ homogeneous, algebraically independent invariants $f_1, f_2, \ldots, f_n \in \mathbb{R}[\underline{x}_n]^G_+$ such that
$$\mathbb{R}[\underline{x}_n]^G = \mathbb{R}[f_1, f_2, \ldots, f_n].$$

① $\Rightarrow$ ② Chevalley's Theorem

② $\Rightarrow$ ① Shephard-Todd Theorem

<u>Working</u> <u>Towards</u> <u>Chevalley</u>...

<u>Lemma</u> Suppose $\lambda \in \mathbb{R}[\underline{x}_n]_1 - 0$ is a <u>nonzero</u> <u>linear</u> polynomial & let $H = \{v \in \mathbb{R}^n : \lambda(v) = 0\}$. Suppose $f \in \mathbb{R}[\underline{x}_n]$ <u>vanishes</u> on H. Then $f = \lambda \cdot g$ for some $g \in \mathbb{R}[\underline{x}_n]$.

<u>Pf</u> WLOG $\lambda = x_1$. Write $f(x_1, \ldots, x_n) = x_1 \cdot g(x_1, \ldots, x_n) + r(x_2, \ldots, x_n)$. If f vanishes on h, so does r. But (b/c $\mathbb{R}$ is <u>infinite</u>) this forces $r = 0$ in $\mathbb{R}[\underline{x}_n]$. //

__Strange__ __Lemma__ Let $G \subseteq GL_n(\mathbb{R})$ be a refl'n gp & consider $I = \langle \mathbb{R}[\underline{x}_n]^G_+ \rangle_{\mathbb{R}[\underline{x}_n]} \subsetneq \mathbb{R}[\underline{x}_n]$. Suppose we have a relation:

$$\circledast \quad p_1 f_1 + p_2 f_2 + \cdots + p_t f_t = 0,$$

where
- $p_1, p_2, \ldots, p_t \in \mathbb{R}[\underline{x}_n]$ are __homogeneous__,
- $f_1, f_2, \ldots, f_t \in \mathbb{R}[\underline{x}_n]^G$ are __invariants__, &
- f_1 does not lie in $\langle f_2, \ldots, f_t \rangle_{\mathbb{R}[\underline{x}_n]^G} \subsetneq \mathbb{R}[\underline{x}_n]^G$.

Then $p_1 \in I = \langle \mathbb{R}[\underline{x}_n]^G_+ \rangle$.

__Pf__ We induct on $\deg p_1$. If $\deg p_1 = 0$, then $p_1 \in \mathbb{R}$ and we can apply R_G to $\circledast$ to get:

$$p_1 \cdot f_1 = -R_G(p_2) \cdot f_2 - \cdots - R_G(p_t) \cdot f_t \in \langle f_2, \ldots, f_t \rangle_{\mathbb{R}[\underline{x}_n]^G}$$

forcing $p_1 = 0 \in I$.

Suppose $\deg p_1 > 0$. Let H be a reflecting hyperplane for G & let s = refl'n thru H. Applying s to $\circledast$ gives:

$$(**) \quad (s.p_1) f_1 + \cdots + (s.p_t) f_t = 0.$$

Subtracting $(**)$ from $\circledast$:

$$(***) \quad (p_1 - s p_1) f_1 + \cdots + (p_t - s.p_t) f_t = 0.$$

But $p_i - s.p_i$ __vanishes__ on H for $i = 1, 2, \ldots, t$ so λ = linear form defining H can be __concelled__ from $(***)$. By induction $\frac{p_1 - s.p_1}{\lambda} \in I$, so $p_1 - s.p_1 \in I$, or $p_1 \equiv s.p_1$ (modulo I).

Since G is generated by reflns,

$$p_1 \equiv g.p_1 \pmod{I} \quad \text{for all } g \in G.$$

Thus

$$R_G(p_1) = \frac{1}{|G|}\sum_{g\in G}(g.p_1) \equiv p_1 \pmod{I}.$$

Since $\deg p_1 > 0$ this means $p_1 \equiv 0 \pmod{I}$ so $p_1 \in I$. $\blacksquare$

Euler's Lemma Suppose $f(x_1,\ldots,x_n) \in \mathbb{R}[x_1,\ldots,x_n]_d$ is homogeneous of degree d. Then

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = d \cdot f.$$

Rmk $\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$ is the "Euler form".

Chevalley's Theorem Let $G \subseteq GL_n(\mathbb{R})$ be a reflection group. There exist n algebraically independent homogeneous polynomials of positive degree $f_1, f_2, \ldots, f_n \in \mathbb{R}[\underline{x}_n]^G$ st

$$\mathbb{R}[x_1,\ldots,x_n]^G = \mathbb{R}[f_1, f_2, \ldots, f_n].$$

Pf Consider $I = \left\langle \mathbb{R}[\underline{x}_n]^G_+ \right\rangle \subseteq \mathbb{R}[\underline{x}_n]$. Let $f_1, f_2, \ldots, f_r \in \mathbb{R}[\underline{x}_n]^G_+$ be homogeneous s.t. $I = \langle f_1, \ldots, f_r\rangle$ & $I \neq \langle f_1, \ldots, \hat{f_i}, \ldots, f_r\rangle$ for any $1 \le i \le r$. Then $\mathbb{R}[\underline{x}_n]^G = \mathbb{R}[f_1,\ldots,f_r]$ (Pf of Hilbert's Thm!) so ~~that~~ we show

$\star$ $\{f_1,\ldots,f_r\}$ is algebraically independent.

Then the "Mdien Corollaries" imply $r = n$.

(***) $\cancel{\sum_{j=0}^{m}\left[h_j + \sum_{p=m+1}^{r}\right.}$

$$0 = \sum_{j=1}^{m} h_j \frac{\partial f_j}{\partial x_i} + \sum_{j=m+1}^{r} \left[\sum_{\ell=1}^{m} g_{j\ell} \cdot h_\ell\right] \frac{\partial f_j}{\partial x_i}, \quad \text{or}$$

(†) $$0 = \sum_{j=1}^{m} h_j \underbrace{\left[\overset{d_j - 1}{\frac{\partial f_j}{\partial x_i}} + \sum_{q=m+1}^{r} \overset{d_j - d_q}{g_{qj}} \overset{d_q - 1}{\frac{\partial f_q}{\partial x_i}} \right]}_{\text{homog deg } d_j - 1.}$$

By Strange Lemma,

$$\frac{\partial f_1}{\partial x_i} + \sum_{q=m+1}^{r} g_{q1} \frac{\partial f_q}{\partial x_i} \in I, \quad \text{so}$$

$\exists \quad p_1, \dots, p_r \in \mathbb{R}[\underline{x}_n]$ homog st

(††) $$\frac{\partial f_1}{\partial x_i} + \sum_{q=m+1}^{r} g_{q,1} \frac{\partial f_q}{\partial x_i} = p_1 f_1 + \cdots + p_r f_r .$$

Multiply (**) by x_i & sum over $1 \le i \le n$. By Euler,

$$d_1 f_1 + \sum_{q=m+1}^{r} d_q \cdot g_{q,1} \cdot f_q = \varphi_1 f_1 + \cdots + \varphi_r f_r, \quad \deg \varphi_i > 0 .$$

B/c d_1 is a constant, $\varphi_1 f_1$ must <u>cancel</u> w/ other terms;

get $f_1 \in \langle f_2, \dots, f_r \rangle$. $\divideontimes$

Let $d_i = \deg f_i$ for $1 \leq i \leq r$, so $d_i > 0$.

If ☆ failed, $\exists$ $0 \neq h(y_1, \dots, y_r) \in \mathbb{R}[y_1, \dots, y_r]$

s.t. $(*)$ $h(f_1, \dots, f_r) = 0$.

Given any monomial $a\, y_1^{a_1} \cdots y_r^{a_r}$ appearing in h,

$f_1^{a_1} \cdots f_r^{a_r}$ is homog of deg. $d = a_1 d_1 + \cdots + a_r d_r$;

WLOG d is <u>constant</u> over all monomials appearing in h.

Given $1 \leq i \leq n$, apply $\frac{\partial}{\partial x_i}$ to $(*)$:

$$(**) \quad 0 = \frac{\partial\, h(f_1, \dots, f_r)}{\partial x_i} \underset{\substack{\uparrow \\ \text{CHAIN} \\ \text{RULE}}}{=} \sum_{j=1}^{r} \overbrace{\frac{\partial h}{\partial y_j}(f_1, \dots, f_r)}^{\substack{h_j \\ \parallel}} \cdot \frac{\partial f_j}{\partial x_i}$$

$$= \sum_{j=1}^{r} \underbrace{h_j}_{\substack{\uparrow \\ \text{homogeneous } \deg = d - d_j \\ \underline{\text{invariant}}}} \cdot \underbrace{\frac{\partial f_j}{\partial x_i}}_{\substack{\nwarrow \\ \text{homogeneous,} \\ \deg = d_j - 1}}$$

To apply <u>Strange Lemma</u>, renumber $h_1, h_2, \dots, h_r$ if necessary so that $\{h_1, \dots, h_m\}$ is a minimal generating set of $\langle h_1, \dots, h_r \rangle_{\mathbb{R}[\underline{x}_n]^G}$.

So for $m < p \leq r$, $\underset{\substack{\uparrow \\ d - d_p}}{h_p} = \sum_{\ell=1}^{m} \underbrace{g_{p\ell}}_{\mathbb{R}[\underline{x}_n]^G_{d_\ell - d_p}} \cdot \overset{d - d_\ell}{h_\ell}$ for some

$g_{p\ell} \in \mathbb{R}[\underline{x}_n]^G_{d_\ell - d_p}$. Now $(**)$ reads:

Math 202C | Lecture 18 | 5/18/18

LAST TIME

Strange Lemma $G \subseteq GL_n(\mathbb{R})$ refl'n group. $I = \langle \mathbb{R}[\underline{x}_n]^G_+ \rangle \subseteq \mathbb{R}[\underline{x}_n]$

Suppose $f_1 h_1 + f_2 h_2 + \cdots + f_t h_t = 0$ where...

- $f_1, f_2, \ldots, f_t \in \mathbb{R}[\underline{x}_n]$ are homogeneous,
- $h_1, h_2, \ldots, h_t \in \mathbb{R}[\underline{x}_n]^G$ are invariant,
- $h_1 \notin \langle h_2, \ldots, h_t \rangle \subseteq \mathbb{R}[\underline{x}_n]^G$.

Then $f_1 \in I$.

Euler Identity Suppose $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is homog. of degree d. Then $\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = d \cdot f$.

RMK $\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ is the Euler form.

Chevalley's Thm Let $G \subseteq GL_n(\mathbb{R})$ be a refl'n gp. There exist n homogeneous algebraically independent invariants $f_1, \ldots, f_n \in \mathbb{R}[\underline{x}_n]^G$ st $\mathbb{R}[\underline{x}_n]^G = \mathbb{R}[f_1, \ldots, f_n]$.

Pf Let $I = \langle \mathbb{R}[\underline{x}_n]^G_+ \rangle \subseteq \mathbb{R}[\underline{x}_n]$ and let $\{f_1, \ldots, f_r\}$ be a minimal homogeneous generating set of I, $f_i \in \mathbb{R}[\underline{x}]^G_+$.
Then (pf of Hilbert) $\mathbb{R}[\underline{x}_n]^G = \mathbb{R}[f_1, \ldots, f_r]$.
We will prove that $\{f_1, \ldots, f_r\}$ is algebraically indep.
By Molien corollaries, this will force $r = n$.

Let $d_i = \deg f_i$ for $1 \leq i \leq r$. If $\{f_1, \ldots, f_r\}$ is NOT alg. indep, $\exists$ $0 \neq h(y_1, \ldots, y_r) \in \mathbb{R}[y_1, \ldots, y_r]$ such that

$$(*) \quad h(f_1, f_2, \ldots, f_r) = 0 .$$

For any monomial $y_1^{a_1} \cdots y_r^{a_r}$ appearing in h, WLOG.

$d = \deg(f_1^{a_1} \cdots f_r^{a_r}) = a_1 d_1 + \cdots + a_r d_r$; $d = \sum_{i=1}^{r} a_i d_i$

is the same for any such monomial.

Now apply $\frac{\partial}{\partial x_i}$ to both sides of $(*)$ (where $1 \leq i \leq n$);

by the Chain Rule,

$$(**) \quad \sum_{j=1}^{r} \overbrace{\underbrace{\frac{\partial h}{\partial y_j}(f_1, \ldots, f_r)}_{d-d_j}}^{h_j} \cdot \underbrace{\frac{\partial f_j}{\partial x_i}}_{d_j - 1} = 0$$

deg:

We have $h_j \in \mathbb{R}[\underline{x}_n]^G$ for $j = 1, 2, \ldots, r$. Renumber, if necessary, so that $\{h_1, h_2, \ldots, h_s\}$ is a minimal generating set for $\langle h_1, \ldots, h_s, h_{s+1}, \ldots, h_r \rangle_{\mathbb{R}[\underline{x}_n]^G} \subseteq \mathbb{R}[\underline{x}_n]^G$.

For $s+1 \leq j \leq r$ we may write:

$$\underset{\deg\ d-d_j}{h_j} = \sum_{k=1}^{s} \underset{d_k - d_j}{g_{kj}} \cdot \underset{d-d_k}{h_k} \qquad (g_{kj} \in \mathbb{R}[\underline{x}_n]^G \text{ homog.})$$

Plugging this into $(**)$ yields:

* If $G \subseteq GL_n(\mathbb{R})$ is a refl'n group, then

$$\mathbb{R}[\underline{x}_n]^G = \mathbb{R}[\overbrace{f_1, f_2, \_, f_n}^{\text{alg. indep., homog.}}]$$

deg: $\quad d_1 \quad d_2 \quad\quad d_n$

$f_1, f_2, \ldots, f_n$: not uniquely determined

$d_1, d_2, \ldots, d_n$: UNIQELY DETERMINED ("degrees" of G)

Ex ① $G = S_n \subseteq GL_n(\mathbb{R})$ degs: $1, 2, \ldots, n$

② $G = B_n/C_n \subseteq GL_n(\mathbb{R})$ degs: $2, 4, \ldots, 2n$

③ $G = I_2(m) \subseteq GL_2(\mathbb{R})$ degs: $2, m$

④ $G = D_n = \left\{ \begin{array}{l} n \times n \text{ signed permutation} \\ \text{matrices w/ even \# of } -1\text{s} \end{array} \right\}$ $\begin{pmatrix} & -1 & \\ & & 1 \\ -1 & & \end{pmatrix}$

Invariants:

$$p_2 = x_1^2 + \cdots + x_n^2$$
$$p_4 = x_1^4 + \cdots + x_n^4$$
$$\vdots$$
$$p_{2n-2} = x_1^{2n-2} + \cdots + x_n^{2n-2}$$
$$x_1 \cdots x_n$$

degs: $2, 4, \ldots, 2n-2, n$.

Recall $|G| = d_1 d_2 \cdots d_n$.

$$d_1 + d_2 + \cdots + d_n = ?$$

THM Let $G \subseteq GL_n(\mathbb{R})$ be a refl'n gp w/ degrees $d_1, \ldots, d_n$.

Then $d_1 + d_2 + \cdots + d_n = n + N$, where

N = # of reflections in G.

$$(***) \quad \sum_{j=1}^{s} h_j \cdot \underbrace{\left[\frac{\partial f_j}{\partial x_i} + \sum_{k=s+1}^{r} g_{jk} \frac{\partial f_k}{\partial x_i} \right]}_{p_j} = 0$$

Applying the Strange Lemma to $h_1 p_1 + \cdots + h_s p_s = 0$, we get $p_1 \in I$, so $\exists$ $q_1, \ldots, q_r \in \mathbb{R}[\underline{x}_n]$ st $I = \langle f_1, \ldots, f_r \rangle$

$$\underbrace{\frac{\partial f_1}{\partial x_1} + \sum_{k=s+1}^{r} g_{1k} \frac{\partial f_k}{\partial x_i}}_{d_1 - 1} = q_1 \cdot f_1 + q_2 \cdot f_2 + \cdots + q_r f_r .$$

deg: $d_1 - 1$ deg d_1 d_2 d_r

WLOG the q_i are homogeneous of deg $d_1 - 1 - d_i$, i.e.

$\boxed{q_1 = 0}$ so that:

$$(\dagger) \quad \frac{\partial f_1}{\partial x_1} + \sum_{k=s+1}^{r} g_{1k} \frac{\partial f_k}{\partial x_i} = q_2 \cdot f_2 + \cdots + q_r \cdot f_r .$$

Now multiply both sides of $(\dagger)$ by x_i & sum the relns over $i = 1, 2, \ldots, n$. This gives an equation of the form (Euler's Identity!)

$$(\dagger\dagger) \quad d_1 \cdot f_1 + \sum_{k=s+1}^{r} d_k \cdot g_{1,k} \cdot f_k = \tilde{q}_2 f_2 + \cdots + \tilde{q}_r \cdot f_r .$$

Since $d_1 \neq 0$, $(\dagger\dagger)$ implies $f_1 \in \langle f_2, f_3, \ldots, f_r \rangle_{\mathbb{R}[\underline{x}_n]}$, contradicting the minimality of $\{f_1, f_2, \ldots, f_r\}$. ∎

Ex ① $S_n \subseteq GL_n(\mathbb{R})$ degs $=$ $1, 2, \ldots, n$

reflns $\longleftrightarrow$ $\underset{1 \leq i < j \leq n}{(i,j)}$ $\quad n + \binom{n}{2} \stackrel{?}{=} 1 + 2 + \cdots + n$ ✓

② $B_n/C_n \overset{\subseteq GL_n(\mathbb{R})}{}$ degs: $2, 4, \ldots, 2n$

$$\begin{array}{lll} x_i + x_j = 0 & 1 \leq i < j \leq n & \binom{n}{2} \\ x_i - x_j = 0 & 1 \leq i < j \leq n & \binom{n}{2} \\ x_i = 0 & 1 \leq i \leq n & n \end{array}$$

$$n + \overset{\substack{N \\ \downarrow}}{\left\{ \binom{n}{2} + \binom{n}{2} + n \right\}} \stackrel{?}{=} 2 + 4 + \cdots + 2n. \ ✓$$

Pf By Molien's Theorem,

$$(*) \quad \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I - tg)} = \prod_{i=1}^{n} \frac{1}{(1 - t^{d_i})}.$$

Recall that $g \in G$ is a refl'n $\iff$ g has eigenvalues $-1, \overbrace{1, \ldots, 1}^{n-1}$.

Multiply both sides of $(*)$ by $(1-t)^n$:

$$(**) \quad \frac{1}{|G|} \cdot \Big\{ \underset{\substack{\uparrow \\ g = I}}{1} + \underset{\substack{\uparrow \\ g \text{ is a refl'n}}}{N \cdot \frac{1-t}{1+t}} + \underset{\substack{\uparrow \\ g \in G \text{ NOT a refl'n}}}{(1-t)^2 \cdot p(t)} \Big\} = \prod_{i=1}^{n} \overset{\frac{1}{1+t+\cdots+t^{d_i-1}}}{\cancel{\frac{1-t}{1-t^{d_i}}}}.$$

Here $p(t)$ is a rational fcn s.t. $p(1)$ is well-defined.

Apply $\frac{\partial}{\partial t}$ to both sides of $(**)$:

$$\frac{1}{|G|} \cdot \Big\{ N \cdot \underbrace{\frac{-(1+t) - (1-t)}{(1+t)^2}}_{\frac{-2}{(1+t)^2}} + 2 \cdot (1-t) p(t) + (1-t)^2 p'(t) \Big\}$$

$$= \left[\prod_{i=1}^{n} \frac{1}{1 + t + \cdots + t^{d_i - 1}} \right] \times \sum_{j=1}^{n} \cancel{(1 + t + \cdots + t^{d_j - 1})} \frac{-1 - 2t - \cdots - (d_j - 1) t^{d_j - 2}}{(1 + t + \cdots + t^{d_j - 1})^{\cancel{2}}}.$$