Problem 1: Prove that the Zariski closure of the integer lattice \( \mathbb{Z}^n \) inside affine complex \( n \)-space \( \mathbb{C}^n \) is \( \mathbb{C}^n \) itself.

Problem 2: Let \( R \) be a ring. Show that the following are equivalent.

1. Any ascending chain of ideals \( I_1 \subseteq I_2 \subseteq \cdots \) in \( R \) stabilizes (i.e., \( R \) is Noetherian).
2. Any nonempty family \( \Sigma \) of ideals in \( R \) contains a maximal element under inclusion.
3. Any ideal of \( R \) is finitely generated.

Problem 3: Let \( R \) be a ring and let \( I, J \subseteq R \) be ideals. The ideal quotient (or colon ideal) \( (I : J) \) is
\[
(I : J) := \{ f \in R : fJ \subseteq I \}.
\]
Prove that \( (I : J) \) is an ideal of \( R \) containing \( I \).

Problem 4: Let \( R \) be a commutative ring and let \( I \subseteq R \) be an ideal. The radical \( \sqrt{I} \) of \( I \) is
\[
\sqrt{I} := \{ f \in R : f^n \in I \text{ for some } n > 0 \}.
\]
Prove that \( \sqrt{I} \) is an ideal containing \( I \).

An ideal \( I \) is called radical if \( I = \sqrt{I} \). Prove that \( I(X) \subseteq \mathbb{K}[x_1, \ldots, x_n] \) is radical for any subset \( X \subseteq \mathbb{K}^n \), where \( \mathbb{K} \) is a field.

Problem 6: Let \( k \) be a field and consider the polynomial ring \( k[x, y] \) in two variables. Let \( R = k[x, xy, xy^2, xy^3, \ldots] \) be the subring of \( k[x, y] \) generated by \( \{xy^n : n \geq 0 \} \). Prove that \( R \) is not Noetherian. (So subrings of Noetherian rings are not necessarily Noetherian.)

Problem 7: Consider the polynomials \( f, f_1, f_2 \in \mathbb{Q}[x, y, z] \) given by
\[
f := x^3 - x^2 y - x^2 z + x \\
f_1 := x^2 y - z \\
f_2 := xy - 1.
\]
Let \( r_1 \) be the remainder of \( f \) upon division by \( (f_1, f_2) \) and \( r_2 \) be the remainder of \( f \) upon division by \( (f_2, f_1) \) using \( <_{\text{grlex}} \).

1. Compute \( r_1 \) and \( r_2 \).
2. Let \( I = \langle f_1, f_2 \rangle \) be the ideal generated by \( f_1 \) and \( f_2 \). Find a nonzero element of \( I \) whose remainder upon division by \( (f_1, f_2) \) is itself.