Math 202C: Spring 2018
Homework 1
Due 4/20/2018

Problem 1: Prove that the Zariski closure of the integer lattice $\mathbb{Z}^n$ inside affine complex n-space $\mathbb{C}^n$ is $\mathbb{C}^n$ itself.

Problem 2: Let R be a ring. Show that the following are equivalent.

(1) Any ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$ in R stabilizes (i.e., R is Noetherian).
(2) Any nonempty family Σ of ideals in R contains a maximal element under inclusion.
(3) Any ideal of R is finitely generated.

Problem 3: Let R be a ring and let $I, J \subseteq R$ be ideals. The *ideal quotient* (or *colon ideal*) $(I : J)$ is

$$(I : J) := \{f \in R \,:\, fJ \subseteq I\}.$$

Prove that $(I : J)$ is an ideal of R containing I.

Problem 4: Let R be a commutative ring and let $I \subseteq R$ be an ideal. The *radical* $\sqrt{I}$ of I is

$$\sqrt{I} := \{f \in R \,:\, f^n \in I \text{ for some } n > 0\}.$$

Prove that $\sqrt{I}$ is an ideal containing I.

An ideal I is called *radical* if $I = \sqrt{I}$. Prove that $\mathbf{I}(X) \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is radical for any subset $X \subseteq \mathbb{K}^n$, where $\mathbb{K}$ is a field.

Problem 6: Let k be a field and consider the polynomial ring $k[x, y]$ in two variables. Let $R = k[x, xy, xy^2, xy^3, \ldots]$ be the subring of $k[x, y]$ generated by $\{xy^n \,:\, n \geq 0\}$. Prove that R is not Noetherian. (So subrings of Noetherian rings are not necessarily Notherian.)

Problem 7: Consider the polynomials $f, f_1, f_2 \in \mathbb{Q}[x, y, z]$ given by

$$f := x^3 - x^2y - x^2z + x$$
$$f_1 := x^2y - z$$
$$f_2 := xy - 1.$$

Let r_1 be the remainder of f upon division by (f_1, f_2) and r_2 be the remainder of f upon division by (f_2, f_1) using $<_{grlex}$.

(1) Compute r_1 and r_2.
(2) Let $I = \langle f_1, f_2 \rangle$ be the ideal generated by f_1 and f_2. Find a nonzero element of I whose remainder upon division by (f_1, f_2) is itself.