Math 202C: Spring 2018
Homework 2
Due 5/4/2018

Problem 1: Fix a monomial order $<$ on monomials in $k[x_1, \ldots, x_n]$, let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal, and let $G = \{g_1, \ldots, g_s\}$ be a Gröbner basis of I. The quotient ring $k[x_1, \ldots, x_n]/I$ is a k-vector space. Prove that the set of monomials

$$\{\text{monomials } m \text{ in } k[x_1, \ldots, x_n] \,:\, \mathrm{LM}(g_i) \nmid m \text{ for } 1 \leq i \leq s\}$$

descends to a k-vector space basis for $k[x_1, \ldots, x_n]/I$.

This is called the *standard monomial basis* for $k[x_1, \ldots, x_n]/I$ with respect to the monomial order $<$.

Problem 2: (202 Qual, Spring 2013) Consider the equations

$$\begin{cases} x^2 - xy - 2x = 0 \\ y^2 - 2xy - y = 0 \end{cases}$$

and let I be the ideal in $\mathbb{C}[x, y]$ generated by these equations.

(1) Find the reduced Gröbner basis for I with respect to the lexicographic order where $y > x$.
(2) Find the reduced Gröbner basis for $I \cap \mathbb{C}[x]$.
(3) Find all solutions to these equations which lie in $\mathbb{C}^2$.
(4) Find a $\mathbb{C}$-vector space basis for $\mathbb{C}[x, y]/I$.

Problem 3: (202 Qual, Spring 2013) Let S be the parametric surface defined by

$$\begin{cases} x = u - 2v \\ y = uv \\ z = v \end{cases}$$

(1) Compute the reduced Gröbner basis for the ideal generated by these equations with respect to the lexicographic order where $u > v > x > y > z$.
(2) Find the equation of the smallest variety V containing S.
(3) Show that $S = V$.

Problem 4: Given a set of variables $\{x_1, x_2, \ldots, x_r\}$, the *degree d complete homogeneous symmetric polynomial* is

$$h_d(x_1, x_2, \ldots, x_r) := \sum_{1 \leq i_1 \leq \cdots \leq i_d \leq r} x_{i_1} x_{i_2} \cdots x_{i_d}.$$

For example,

$$h_2(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2.$$

By convention, $h_0(x_1, \ldots, x_r) = 1$.

(1) Prove the polynomial identity

$$h_d(x_1, x_2, \ldots, x_{i-1}, x_i) = x_i \cdot h_{d-1}(x_1, x_2, \ldots, x_{i-1}, x_i) + h_d(x_1, x_2, \ldots, x_{i-1}),$$

valid for any $d \geq 1$ and $i \geq 2$.

(2) The *invariant ideal* I_n is the ideal in $\mathbb{C}[x_1, x_2, \ldots, x_n]$ generated by

$$h_1(x_1, x_2, \ldots, x_n), h_2(x_1, x_2, \ldots, x_n), \ldots, h_n(x_1, x_2, \ldots, x_n).$$

Prove that, for $1 \leq i \leq n$, we have

$$h_i(x_1, x_2, \ldots, x_{n-i+1}) \in I_n.$$

(3) A monomial $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ is called *sub-staircase* if $0 \leq a_i \leq n - i$ for all i. Prove that the $n!$ sub-staircase monomials descend to a $\mathbb{C}$-vector space spanning set for the *coinvariant ring* $R_n := \mathbb{C}[x_1, \ldots, x_n]/I_n$. [1]

(Hint: Consider the lexicographical order where $x_n > x_{n-1} > \cdots > x_1$. What do you know about the leading term ideal of I_n?)

[1]Actually, these monomials are the standard monomial basis for this quotient, but this is a bit trickier to prove. The algebraic properties of R_n are deeply tied to the combinatorial properties of permutations in S_n.