Math 202C: Spring 2018
Homework 3
Due 5/18/2018

Problem 1: Let k be an algebraically closed field, let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal, and let $V = \mathbf{V}(I) \subseteq k^n$ be the variety corresponding to I. Prove that the quotient ring $k[x_1, \ldots, x_n]/I$ is finite-dimensional as a k-vector space if and only if V is a finite set. What happens if k is not algebraically closed.

Problem 2: (202 Qual, Spring 2013) Let k be an algebraically closed field. Two ideals $I, J \subseteq k[x_1, \ldots, x_n]$ are *comaximal* if $I + J = k[x_1, \ldots, x_n]$. Prove that I and J are comaximal if and only if $\mathbf{V}(I) \cap \mathbf{V}(J) = \varnothing$. What happens when k is not algebraically closed?

Problem 3: Consider the matrix $A \in GL_3(\mathbb{Q})$ given by

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let G be the cyclic group generate by A. Use Molien's Theorem to find the Hilbert series of the invariant ring $\mathbb{Q}[x, y, z]^G$.

Problem 4: Find two order 3 subgroups $H, K \subset GL_3(\mathbb{C})$ such that the invariant rings $\mathbb{C}[x, y, z]^H$ and $\mathbb{C}[x, y, z]^K$ are *not* isomorphic as graded algebras. (Note that H and K must be isomorphic as abstract groups!)

Problem 5: Let $\zeta = \exp(2\pi i/3)$ and let $A, B \in GL_2(\mathbb{C})$ be the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}.$$

Let G be the subgroup of $GL_2(\mathbb{C})$ generated by A and B.

(a) Find the Hilbert series of $\mathbb{C}[x, y]^G$.

(b) Do there exist algebraically independent homogeneous invariants $f_1, f_2 \in \mathbb{C}[x, y]^G$ such that

$$\mathbb{C}[x, y]^G = \mathbb{C}[f_1, f_2]?$$

Either find such invariants, or prove that they do not exist.