Problem 1: Let $k$ be an algebraically closed field, let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal, and let $V = V(I) \subseteq k^n$ be the variety corresponding to $I$. Prove that the quotient ring $k[x_1, \ldots, x_n]/I$ is finite-dimensional as a $k$-vector space if and only if $V$ is a finite set. What happens if $k$ is not algebraically closed.

Problem 2: (202 Qual, Spring 2013) Let $k$ be an algebraically closed field. Two ideals $I, J \subseteq k[x_1, \ldots, x_n]$ are comaximal if $I + J = k[x_1, \ldots, x_n]$. Prove that $I$ and $J$ are comaximal if and only if $V(I) \cap V(J) = \emptyset$. What happens when $k$ is not algebraically closed?

Problem 3: Consider the matrix $A \in GL_3(\mathbb{Q})$ given by

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$  

Let $G$ be the cyclic group generated by $A$. Use Molien’s Theorem to find the Hilbert series of the invariant ring $\mathbb{Q}[x,y,z]^G$.

Problem 4: Find two order 3 subgroups $H, K \subset GL_3(\mathbb{C})$ such that the invariant rings $\mathbb{C}[x,y,z]^H$ and $\mathbb{C}[x,y,z]^K$ are not isomorphic as graded algebras. (Note that $H$ and $K$ must be isomorphic as abstract groups!)

Problem 5: Let $\zeta = \exp(2\pi i/3)$ and let $A, B \in GL_2(\mathbb{C})$ be the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}.$$  

Let $G$ be the subgroup of $GL_2(\mathbb{C})$ generated by $A$ and $B$.

(a) Find the Hilbert series of $\mathbb{C}[x,y]^G$.

(b) Do there exist algebraically independent homogeneous invariants $f_1, f_2 \in \mathbb{C}[x,y]^G$ such that $\mathbb{C}[x,y]^G = \mathbb{C}[f_1, f_2]$? Either find such invariants, or prove that they do not exist.