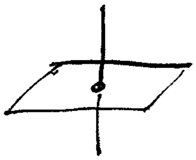
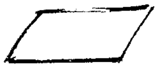
Math 202C Lecture 12 5/4/18

LAST TIME **Forms of Nullstellensatz:**

$\underline{k = \bar{k}}$ We have inverse maps...

$$\begin{tikzcd}
\left\{ \begin{matrix} \text{radical ideals} \\ I \subseteq k[\underline{x}_n] \end{matrix} \right\} \arrow[r, bend left, "\mathbb{V}"] \arrow[d, bend right, "\text{restricts to}"'] & \left\{ \begin{matrix} \text{varieties} \\ V \subseteq k^n \end{matrix} \right\} \arrow[l, bend left, "\mathbb{I}"] \\
\left\{ \begin{matrix} \text{prime ideals} \\ P \subseteq k[\underline{x}_n] \end{matrix} \right\} \arrow[r, bend left, "\mathbb{V}"] \arrow[d, bend right, "\text{restricts to}"'] & \left\{ \begin{matrix} \text{irreducible} \\ \text{varieties} \\ V \subseteq k^n \end{matrix} \right\} \arrow[l, bend left, "\mathbb{I}"] \\
\left\{ \begin{matrix} \text{maximal ideals} \\ M \subseteq k[\underline{x}_n] \end{matrix} \right\} \arrow[r, bend left, "\mathbb{V}"] & \left\{ \begin{matrix} \text{points} \\ (a_1, -, a_n) \in k^n \end{matrix} \right\} \arrow[l, bend left, "\mathbb{I}"] \\
\langle x_1 - a_1, —, x_n - a_n \rangle \arrow[r, leftrightarrow] & (a_1, —, a_n)
\end{tikzcd}$$





§7 Invariant Theory of Finite Groups

* Assume char $k = 0$, so that $|k| = \infty$. *

- $GL_n(k)$ acts on k^n: $\underset{GL_n k}{\underset{\cap}{A}} \cdot \underset{k^n}{\underset{\cap}{v}} = \underbrace{Av}_{\text{matrix mult.}}$

- **Identification** $k[x_1, -, x_n] = \left\{ \text{all polynomial fcns } \overset{f:}{k^n} \to k \right\}$.

eg $n=3$ $\quad x_1^2 x_2 - 2x_2x_3 : \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \mapsto 3^2 \cdot 2 - 2 \cdot 2 \cdot 1 = 14.$

So $GL_n(k) \circlearrowright k[x_1, —, x_n]$ via:

$$(\underset{GL_n(k)}{\underset{\cap}{A}} \cdot \underset{k[\underline{x}_n]}{\underset{\cap}{f}})(\underset{k^n}{\underset{\cap}{v}}) := f(A^{-1}v).$$

$$\ulcorner \quad I.f = f \quad , \quad (AB).f = A.(B.f) \quad \lrcorner$$

$$(AB).f(v) = f((AB)^{-1}v) = f(B^{-1}A^{-1}v) = (B.f)(A^{-1}v)$$
$$= A.(B.f)(v)$$

~~[illegible]~~ In coordinates

$GL_n(k)$ has 3 kinds of Generators:

$$\begin{array}{c|c} & i \\ \hline i & \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & c & \\ & & & \ddots \\ & & & & 1 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} i & i+1 \end{array} \\ \begin{array}{c} i \\ i+1 \end{array} & \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} & i \\ \hline i & \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & a \\ & & & 1 \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix} \end{array}$$

$c \in k^\times$, $1 \leq i \leq n$ $\qquad$ $1 \leq i \leq n-1$ $\qquad$ $1 \leq i \leq n-1$, $a \in k$

<u>Rules</u>

$$\begin{cases} A.(f+g) = A.f + A.g & \text{for all } A \in GL_n(k) \\ & f, g \in k[\underline{x}_n] \\ A.(f \cdot g) = (A.f) \cdot (A.g) \end{cases}$$

$$\begin{cases} (AB).f = A.(B.f) & \text{for all } A, B \in GL_n(k) \\ I.f = f & f \in k[\underline{x}_n] \end{cases}$$

$$A.c = c \qquad \text{for all } c \in k$$

$$\begin{array}{c|c} & i \\ \hline i & \begin{bmatrix} 1 & & \\ & c & \\ & & \ddots \\ & & & 1 \end{bmatrix} \end{array} . x_j = \begin{cases} x_j & j \neq i \\ c^{-1}x_i & j = i \end{cases} \qquad \begin{array}{c|c} & i \\ \hline i & \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \end{array} . x_j = \begin{cases} x_j & j \neq i, i+1 \\ x_{i+1} & j = i \\ x_i & j = i+1 \end{cases}$$

$$\begin{array}{c|c} & i \\ \hline i & \begin{bmatrix} 1 & & & \\ & 1 & a & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix} \end{array} . x_j = \begin{cases} x_j & j \neq i \\ x_i - a x_{i+1} & j = i \,. \end{cases}$$

In general

$$k[x_1,\dots,x_n]^{S_n} = k[e_1, -, e_n].$$

Rmk · $k[x_1,-,x_n]^G$ is a subring of $k[x_1,-,x_n]$. for any $G \subseteq GL_n(k)$.

· $k \subseteq k[x_1,-,x_n]^G$.

FACT - If $G = \{I\}$, $k[x_1,-,x_n]^{\{I\}} = k[x_1,-,x_n]$

- $k[x_1,-,x_n]^{GL_n(k)} = k$.

- $H \subseteq G \Rightarrow k[x_1,-,x_n]^H \supseteq k[x_1,-,x_n]^G$.

Ex $k = \mathbb{C}$, $n = 1$, $\zeta = e^{2\pi i/r} \in \mathbb{C}$.

$G = \{(1),(\zeta),\dots,(\zeta^{r-1})\} \subseteq GL_1(\mathbb{C})$; $G \circlearrowleft \mathbb{C}[x]$.

CHECK $\mathbb{C}[x]^G = \mathbb{C}[x^r]$:

KEY: $(\zeta).x = \zeta^{-1}x$; $(\zeta).\,x^r = \overbrace{[(\zeta).x]\cdots[(\zeta).x]}^{r}$

$$= \underbrace{(\zeta^{-1}x)\dots(\zeta^{-1}x)}_{r} = \zeta^{-r}.x^r = x^r.$$

FACT Let $A \in GL_n(k)$, $f \in k[\underline{x}_n]$, & write $f = f_d + \dots + f_1 + f_0$ where f_i is homog. of degree i. Then $A.f = f \iff A.f_i = f_i$ for all $i = 0,1,\dots,d$.

$\Rightarrow$ For $G \subseteq GL_n(k)$, $k[x_1,\dots,x_n]^G$ is a graded k-algebra.

$*$ Any subgroup $G \subseteq GL_n(k)$ acts on $k[x_1, \_, x_n]$:

$$(\underset{\substack{\Cap\\ G}}{A} . \underset{\substack{\Cap\\ k[\underline{x}_n]}}{f})(\underset{\substack{\Cap\\ k^n}}{v}) = f(\underset{\substack{\Cap\\ k^n}}{\underbrace{A^{-1}.v}}).$$

Def Given $G \subseteq GL_n(k)$, the <u>fixed</u> <u>subalgebra</u> of $k[x_1, \_, x_n]$ is

$$k[x_1, \_, x_n]^G = \{f \in k[x_1, \_, x_n] : A.f = f \text{ for all } A \in G\}.$$

Ex $G = \{\text{all } n \times n \text{ permutation matrices}\} \cong S_n$.

$$\Rightarrow k[x_1, \_, x_n]^{S_n} = \{f \in k[x_1, \_, x_n] : f(x_1, \_, x_n) = f(x_{\sigma(1)}, \_, x_{\sigma(n)}) \text{ for all } \sigma \in S_n\}$$

$$= \{\text{all } \underline{\text{symmetric}}\ \underline{\text{polynomials}} \text{ in } x_1, \dots, x_n /k\}.$$

Newton's Thm Let $e_d = e_d(x_1 \dots x_n) = \sum_{1 \le i_1 < \dots < i_d \le n} x_{i_1} \cdots x_{i_d}$. Then <u>any</u> symmetric poly. may be expressed <u>uniquely</u> as a poly. in $e_1, e_2, \dots, e_n$.

eg $\underline{n=3}$

$$e_1 = x_1 + x_2 + x_3$$
$$e_2 = x_1x_2 + x_1x_3 + x_2x_3$$
$$e_3 = x_1x_2x_3$$

$$f = x_1^3 + x_2^3 + x_3^3$$
$$= e_1^3 - 3 \cdot (x_1x_2^2 + x_1^2x_2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2) \bullet 6\underbrace{x_1x_2x_3}_{e_3}$$

$$= e_1^3 \bullet 6e_3 - 3[e_1e_2 - 3 \cdot e_3]$$
$$= e_1^3 - 3e_1e_2 + \overset{3}{\cancel{}}e_3$$

$$\Rightarrow k[\underset{\substack{\uparrow\\1}}{x_1}, \underset{\substack{\uparrow\\1}}{x_2}, \underset{\substack{\uparrow\\1}}{x_3}]^{S_3} \cong k[\underset{\substack{\uparrow\\1}}{e_1}, \underset{\substack{\uparrow\\2}}{e_2}, \underset{\substack{\uparrow\\3}}{e_3}] \left(\cong k[\underset{1}{p_1}, \underset{2}{p_2}, \underset{3}{p_3}] \cong k[\underset{1}{h_1}, \underset{2}{h_2}, \underset{3}{h_3}]\right)$$

deg 1 1 1 deg 1 2 3 deg 1 2 3 1 2 3

Def Let $V = \bigoplus_{d \geq 0} V_d$ be a graded vector space with $\dim V_d < \infty$ for all d.

The Hilbert series is

$$\mathrm{Hilb}(V_d ; t) \overset{\mathrm{def}}{=} \sum_{d \geq 0} \dim(V_d) \cdot t^d .$$

Ex $V = \mathbb{C}[x,y]$.

d	basis	$\dim V_d$
0	1	1
1	x, y	2
2	x^2, xy, y^2	3
3	x^3, x^2y, xy^2, y^3	4
	$\vdots$	
d	$x^d, \dots, xy^{d-1}, y^d$	$d+1$

$$\Rightarrow \mathrm{Hilb}(\mathbb{C}[x,y]; t) = 1 + 2 \cdot t + 3 \cdot t^2 + 4 \cdot t^3 + \cdots$$

$$= \frac{d}{dt}\left[\cancel{1}\ 1 + t + t^2 + t^3 + t^4 + \cdots \right]$$

$$= \frac{d}{dt}\left[\frac{1}{1-t} \right] = \frac{1}{(1-t)^2} .$$

$$\mathrm{Hilb}(\mathbb{C}[x_1, \dots, x_n]; t) = \frac{1}{(1-t)^n} .$$ (CHECK!)

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LAST TIME ☆ char $k = 0$ ☆

$$\underset{GL_n(k)}{\circlearrowright}\; k[x_1, -, x_n] = \{\text{polynomial fcns } f : k^n \to k\}$$

$$(\underset{\substack{\cap \\ GL_n(k)}}{A} . \underset{\substack{\cap \\ k[\underline{x}_n]}}{f})(\underset{\substack{\cap \\ k^n}}{v}) := f(A^{-1}v)$$

$\Rightarrow$ Given $\underset{\text{subgp.}}{G \subseteq} GL_n(k)$, $k[\underline{x}_n]^G = \{f \in k[\underline{x}_n] : A.f = f \text{ for all } A \in G\}$.

Ex $k[\underline{x}_n]^{\{I\}} = k[\underline{x}_n]$; $k[\underline{x}_n]^{GL_n(k)} = k$

$$k[x_1, \dots, x_n]^{S_n} = k[e_1, \dots, e_n] \left(= k[h_1, -, h_n] = k[p_1, -, p_n]\right).$$

Def Let $V = \bigoplus_{d \geq 0} V_d$ be a graded vector space s.t. $\dim V_d < \infty$ for all $d \geq 0$. The *Hilbert series* is

$$\mathrm{Hilb}(V; q) := \sum_{d \geq 0} (\dim V_d) \cdot q^d.$$

Ex $V = \mathbb{C}[x, y]$

d	basis for V_d	$\dim V_d$
0	1	1
1	$x \quad y$	2
2	$x^2 \quad xy \quad y^2$	3
3	$x^3 \quad x^2y \quad xy^2 \quad y^3$	4
d		$d+1$

$$\Rightarrow \mathrm{Hilb}(\mathbb{C}[x,y]; q) = \sum_{d \geq 0} (d+1) \cdot q^d$$

$$= \frac{d}{dq} \sum_{d \geq 0} q^d = \frac{d}{dq}\left(\frac{1}{1-q}\right)$$

$$= \frac{1}{(1-q)^2}.$$

CHECK $\quad \mathrm{Hilb}(\overset{k}{\cancel{\mathbb{C}}}[x_1,\ldots,x_n];q) = \frac{1}{(1-q)^n}.$

Ex $\quad \mathrm{Hilb}(k[x_1,x_2,x_3]^{S_3};q) = \,?$

Well, $k[x_1,x_2,x_3]^{S_3} = k[e_1,e_2,e_3]$ has basis $\{\underbrace{e_1^{a_1}e_2^{a_2}e_3^{a_3}}_{} : a_i \geq 0\}$

$\mathrm{deg} = 1\cdot a_1 + 2\cdot a_2 + 3\cdot a_3$

So $\quad \mathrm{Hilb}(k[x_1,x_2,x_3]^{S_3};q) = \sum_{a_1,a_2,a_3\geq 0} q^{a_1} q^{2\cdot a_2} q^{3\cdot a_3} = \frac{1}{(1-q)(1-q^2)(1-q^3)}$

In general,

$$\mathrm{Hilb}(k[x_1,\ldots,x_n]^{S_n};q) = \prod_{i=1}^{n} \frac{1}{1-q^i}.$$

Ex $\quad G,H \subseteq GL_2(k) \qquad G = \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right\} \quad H = \left\{\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}$

(So $G \cong H \cong C_2$ as <u>abstract groups</u>.)

$$k[x,y]^G \underset{\substack{\uparrow\\ \text{Newton's Thm}}}{=} k[\underbrace{x+y\,,\,xy}_{\substack{\text{alg indep,}\\ \text{homog}}}] \Rightarrow \mathrm{Hilb}(k[x,y]^G;q) = \frac{1}{(1-q)(1-q^2)}$$

$$k[x,y]^H = k[\underbrace{x^2,xy,y^2}_{}] = \bigoplus_{\substack{d\geq 0\\ d\ \underline{\text{even}}}} k[x,y]_d$$

$\begin{pmatrix}-1&\\&-1\end{pmatrix} : \begin{matrix} x\mapsto -x\\ y\mapsto -y\end{matrix} \qquad \underleftarrow{\text{no a.i.h.g.'s}}$

$$\frac{1}{\cancel{2}}\left[\frac{\cancel{2}q^2 + \overset{1}{\cancel{2}}}{(1-q^2)^2}\right].$$

$$\parallel$$

$$\frac{1}{2}\left[\frac{1}{(1-q)^2} + \frac{1}{(1+q)^2}\right]$$

$$\parallel$$

$$\mathrm{Hilb}(k[x,y]^H;q) = \sum_{\substack{d\geq 0\\ d \text{ even}}} (d+1)\cdot q^d = \frac{1}{2}\left[\sum_{d\geq 0}(d+1)q^d + \sum_{d\geq 0}(d+1)(-q)^d\right]$$

* Even though $G \cong H$ as gps,

$$\mathrm{Hilb}(k[x,y]^G; q) \neq \mathrm{Hilb}(k[x,y]^H; q).$$

<u>**FACT**</u> Suppose $G, H \subseteq GL_n(k)$ are s.t. $G = AHA^{-1}$ for some $A \in GL_n(k)$. Then $\mathrm{Hilb}(k[\underline{x}_n]^G; q) = \mathrm{Hilb}(k[\underline{x}_n]^H; q)$.

[In fact, $k[\underline{x}_n]^G \cong k[\underline{x}_n]^H$ as graded k-algebras.]

<u>**Ex**</u> $G = \{ n \times n$ <u>signed</u> perm. matrices$\} \subseteq GL_n(k)$. $\begin{pmatrix} & 1 & \\ & & -1 \\ 1 & & \end{pmatrix}$

$|G| = 2^n \cdot n!$

$k[x_1, \ldots, x_n]^G$ is gen'd by alg. indep

$$\left\{ \begin{array}{ll} & \text{deg} \\ e_1(x_1^2, \ldots, x_n^2) & 2 \\ e_2(x_1^2, \ldots, x_n^2) & 4 \\ \vdots & \vdots \\ e_n(x_1^2, \ldots, x_n^2) & 2n \end{array} \right.$$

$$\Rightarrow \quad \mathrm{Hilb}(k[x_1, \ldots, x_n]^G; q) = \frac{1}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}.$$

<u>**Q**</u> Let $G \subseteq GL_n(k)$ be a finite subgp. Is $k[x_1, \ldots, x_n]^G$ finitely gen'd as a k-algebra? If so, can we give a generating set?

<u>**A**</u> Yes [Hilbert] ; Yes [Noether]

⚠ $$\underbrace{k[x,y]}_{\substack{\text{gen'd as a} \\ k\text{-alg. by } x,y}} \supseteq \underbrace{k[x, xy, xy^2, \ldots]}_{\substack{\underline{\text{not}} \text{ finitely gen'd} \\ \text{as a } k\text{-algebra!}}}$$

__Q__ Given $\underset{\text{finite}}{G \subseteq} GL_n(k)$, can we __calculate__ $\mathrm{Hilb}(k[\underline{x}_n]^G; q)$?

__A__ __Yes__ [Molien].

__Def__ Let $G \subseteq GL_n(k)$ be a finite subgp (__recall__ char $k = 0$).
The __Reynolds__ __operator__ $R_G : k[\underline{x}_n] \longrightarrow k[\underline{x}_n]$ is

$$R_G : f \longmapsto \frac{1}{|G|} \sum_{A \in G} (A.f) . \quad \ulcorner \text{graded ; } k\text{-linear} \lrcorner$$

__FACT__ - $R_G(f) \in k[\underline{x}_n]^G$ for all $f \in k[\underline{x}_n]$
- If $f \in k[\underline{x}_n]^G$, then $R_G(f) = f$.

__Molien's__ __Theorem__ Let $G \subseteq GL_n(k)$ be a finite matrix group.
Then

$$\mathrm{Hilb}(k[\underline{x}_n]^G ; q) = \frac{1}{|G|} \cdot \sum_{A \in G} \frac{1}{\det(I - q \cdot A)} .$$

__Ex__ $G = \left\{ \pm\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \pm\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\} \subseteq GL_2(k)$.

$\underline{A}$	$\underline{\det(I - q \cdot A)}$
$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{vmatrix} 1-q & \\ & 1-q \end{vmatrix} = (1-q)^2$
$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$	$(1+q)^2$
$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$	$\begin{bmatrix} 1 & -q \\ -q & 1 \end{bmatrix}$ $1 - q^2$
$\begin{pmatrix} & -1 \\ -1 & \end{pmatrix}$	$1 - q^2$

$$\Rightarrow \mathrm{Hilb}(k[x_1, x_2]^G ; q)$$

$$= \frac{1}{4}\left[\frac{2}{1-q^2} + \frac{1}{(1-q)^2} + \frac{1}{(1+q)^2} \right]$$

$$= \frac{1}{4}\left[\frac{2[1-q^2] + [1+q]^2 + [1-q]^2}{(1-q)^2(1+q)^2} \right]$$

$$= \frac{1}{4}\left[\frac{4}{(1-q)^2(1+q)^2} \right]$$

$$= \frac{1}{(1-q^2)(1-q^2)} . \quad \ulcorner k[x_1, x_2]^G = k[\underbrace{x_1 x_2, x_1^2 + x_2^2}_{\text{alg indep!}}] \lrcorner .$$

Pf of Molien Since $R_G : k[\underline{x}_n] \twoheadrightarrow k[\underline{x}_n]^G$ is a graded projection,

$$\text{Hilb}\left(k[\underline{x}_n]^G; q\right) = \sum_{d \geq 0} \text{trace}\left(R_{G,d} : k[\underline{x}_n]_d \twoheadrightarrow k[\underline{x}_n]^G_d\right) \cdot q^d$$

(*)

$$= \frac{1}{|G|} \sum_{A \in G} \sum_{d \geq 0} \text{trace}\begin{pmatrix} k[\underline{x}_n]_d \longrightarrow k[\underline{x}_n]_d \\ f \longmapsto A.f \end{pmatrix} q^d .$$

We consider the trace of $\begin{matrix} k[\underline{x}_n]_d \to k[\underline{x}_n]_d \\ f \longmapsto A.f \end{matrix}$. Since $|G| < \infty$, we may diagonalize A (over $\bar{k}$) to write $A \sim \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{pmatrix}$ for some $c_i \in \bar{k}$. For any monomial $x_1^{a_1} \cdots x_n^{a_n}$,

$$\begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix} . \; x_1^{a_1} \text{---} x_n^{a_n} = c_1^{-a_1} \text{---} c_n^{-a_n} \; x_1^{a_1} \text{---} x_n^{a_n} .$$

So,

$$\sum_{d \geq 0} \text{tr}\begin{pmatrix} k[\underline{x}_n]_d \longrightarrow tr[\underline{x}_n]_d \\ f \longmapsto A.f \end{pmatrix} q^d = \sum_{a_1, \ldots, a_n \geq 0} c_1^{-a_1} \cdots c_n^{-a_n} \, q^{a_1} \cdots q^{a_n}$$

$$= \prod_{i=1}^{n} \frac{1}{1 - c_i^{-1} \cdot q} = \frac{1}{\det\begin{pmatrix} 1 - c_1^{-1} q & & \\ & \ddots & \\ & & 1 - c_n^{-1} q \end{pmatrix}}$$

$$= \frac{1}{\det(I - A^{-1} \cdot q)} .$$

So (*) reads $\text{Hilb}\left(k[\underline{x}_n]^G; q\right) = \dfrac{1}{|G|} \displaystyle\sum_{A \in G} \dfrac{1}{\det(I - A^{-1} q)} \underset{\uparrow}{=} \dfrac{1}{|G|} \displaystyle\sum_{A \in G} \dfrac{1}{\det(I - Aq)}$.

$A \in G \Longleftrightarrow A^{-1} \in G$

//

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LAST TIME - If $V = \bigoplus_{d \geq 0} V_d$ is a graded vector space,

$$\mathrm{Hilb}(V; q) = \sum_{d \geq 0} (\dim V_d) \cdot q^d .$$

Molien's Thm (char k = 0). Let $G \subseteq GL_n(k)$ be a finite subgp.

$$\mathrm{Hilb}(k[x_1, -, x_n]^G; q) = \frac{1}{|G|} \cdot \sum_{A \in G} \frac{1}{\det(I - qA)} .$$

Consequences of Molien

Thm Let $G \subseteq GL_n(k)$ be finite and suppose that

$$k[x_1, \ldots, x_n]^G = k[f_1, \ldots, f_r]$$

for some algebraically indep, homogeneous $f_i \in k[\underline{x}_n]^G$.

① $r = n$.

② If $k[x_1, -, x_n]^G = k[g_1, -, g_m]$ for some alg. indep, homogeneous $g_i \in k[\underline{x}_n]^G$ then (up to permutation) $\deg f_i = \deg g_i$ for all i.

③ If $d_i = \deg f_i$ for $i = 1, 2, \ldots, n$ then $|G| = d_1 \cdot d_2 \cdot \ldots \cdot d_n$.

⌜ $d_1, \ldots, d_n$ are the "(invariant) degrees" of G. ⌟

⌜ $k[x_1, -, x_n]^{S_n} = k[e_1, -, e_n] = k[p_1, -, p_n]$

deg $1, -, n$ $\quad$ $1, -, n$

$1 \cdot 2 \cdot \ldots \cdot n = n! = |S_n|$. ⌟

Pf ① By Molien's Thm, if $d_i = \deg f_i$:

$$\mathrm{Hilb}(k[\underline{x}_n]^G; q) = \frac{1}{|G|} \sum_{A \in G} \frac{1}{\det(I - q \cdot A)}$$

⊛

$$\begin{gathered} \| \\ \mathrm{Hilb}(k[f_1, \dots, f_r]; q) = \prod_{i=1}^{r} \frac{1}{1 - q^{d_i}} . \end{gathered}$$

$q = 1$ is a zero of multiplicity 1 in $1 - q^d$ for *any* $d \geq 1$, so $q = 1$ is a pole of $\prod_{i=1}^{r} \frac{1}{(1-q^{d_i})}$ of order r. OTOH, $q = 1$ is a pole of $\frac{1}{\det(I - qA)}$ of order (multiplicity of 1 as an eigenvalue of A). So $q = 1$ is a pole of $\frac{1}{\det(I-qA)}$ of order $\begin{cases} n & A = I \\ < n & A \neq I . \end{cases}$

Thus $q = 1$ is a pole of $\frac{1}{|G|} \sum_{A \in G} \frac{1}{\det(I - qA)}$ of order n. //

② We have

$$\prod_{i=1}^{n} \frac{1}{1 - q^{d_i}} = k[f_1, \dots, f_n] = k[g_1, \dots, g_n] = \prod_{i=1}^{n} \frac{1}{1 - q^{e_i}} .$$

Since $\frac{1}{1-q^d}$ has simple poles at $\left(e^{\frac{2\pi i}{d}}\right)^j$ $(j = 0, 1, \dots, d-1)$ we must have $d_i = e_i$ (up to rearrangement). //

③ We have

$$\circledast \quad \prod_{i=1}^{n} \frac{1}{(1-q^{d_i})} = \frac{1}{|G|} \sum_{A \in G} \frac{1}{\det(I-qA)} .$$

Multiply both sides f $\circledast$ by $(1-q)^n$ and take the limit

as $q \to 1$:

LHS: $$\lim_{q\to1} \prod_{i=1}^{n} \frac{1-q}{1-q^{d_i}} = \lim_{q \to 1} \prod_{i=1}^{n} \frac{1}{1+q+\dots+q^{d_i-1}} = \frac{1}{d_1 d_2 \cdots d_n} .$$

RHS $$\lim_{q\to1} \frac{(1-q)^n}{\det(I-qA)} \underset{\substack{\uparrow \\ A \text{ is } n\times n}}{=} \begin{cases} 0 & \text{if } 1 \text{ is an eVa of } A \text{ of mult} < n \quad (A \neq I) \\ 1 & \text{if } A = I \end{cases}$$

So $$\lim_{q\to1} \frac{1}{|G|} \sum_{A\in G} \frac{1}{\det(I-qA)} = \frac{1}{|G|} \cdot 1 = \frac{1}{|G|} . \quad //$$

Ex ① $G = \{n\times n \text{ signed perm. matrices}\} =: B_n \quad \begin{bmatrix} & -1 & \\ & & 1 \\ -1 & & \end{bmatrix}$

$B_n \circlearrowright k[x_1, \dots, x_n] \qquad |B_n| = 2^n \cdot n!$

Invariants: $\underset{\substack{\| \\ p_2}}{x_1^2 + \dots + x_n^2}, \quad \underset{\substack{\| \\ p_4}}{x_1^4 + \dots + x_n^4}, \quad \dots, \quad \underset{\substack{\| \\ p_{2n}}}{x_1^{2n} + \dots + x_n^{2n}}$

FACT $$k[\underline{x}_n]^{B_n} = k[\overbrace{p_2, p_4, \dots, p_{2n}}^{\text{alg ndep, homog}}]$$

degs: $2, 4, \dots, 2n$

$\Rightarrow \quad 2 \cdot 4 \cdots (2n) = 2^n \cdot n! .$

② $G = D_n = \left\{ \begin{array}{c} n\times n \quad \text{perm matrices}, \\ \text{even } \# \text{ of } -1\text{'s} \end{array} \right\}$ $\qquad \begin{pmatrix} & -1 & \\ & & 1 \\ -1 & & \end{pmatrix}$

$(n \geq 2)$

$$|G| = \frac{2^n \cdot n!}{2} = 2^{n-1} \cdot n!$$

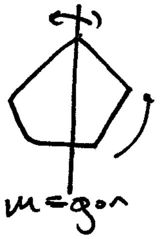
<u>Invariants</u> $\quad p_2, p_4, \dots, p_{2n-2}, \quad (x_1 \cdots x_n)$

$$k[x_1, \_, x_n]^{D_n} = k[\overbrace{p_2, p_4, \dots, p_{2n-2}, \; x_1x_2\cdots x_n}^{\text{alg indep}}]$$

$$\text{deg} \qquad 2 \quad 4 \qquad (2n-2) \qquad n$$

$$2\cdot 4 \cdots (2n-2)\cdot n = 2^{n-1}\cdot n!$$

③ $G = I_2(m) = \left\{ \left\langle \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} \zeta & \\ & \zeta^{-1} \end{pmatrix} \right\rangle \right.,$ $\qquad \zeta = \exp(2\pi i/m)$.



$G \circlearrowleft \mathbb{C}[x,y]$. $\qquad |G| = 2m$

$$\mathbb{C}[x,y]^G = \mathbb{C}[\,xy, \; x^m + y^m\,]$$

$$\text{deg} \qquad 2 \qquad m$$

<u>Hilbert's</u> <u>Theorem</u> Let $G \subseteq GL_n(k)$ be a finite subgp. Then $k[x_1, \_, x_n]^G$ is a finitely-generated k-algebra.

<u>Proof</u> Let $k[\underline{x}_n]^G_+ = \{ f(x_1, \_, x_n) \in k[\underline{x}_n]^G : f(0,..,0) = 0 \}$ & let $I = \langle k[\underline{x}_n]^G_+ \rangle \subseteq k[\underline{x}_n]$. Then I is gen'd by <u>homogeneous</u> invariants $f \in k[\underline{x}_n]^G$ st $\deg f > 0$.

By <u>Hilbert's</u> <u>Basis</u> <u>Theorem</u>, $\exists$ <u>finitely</u> <u>many</u> homog. $f_1, \_, f_r \in k[\underline{x}_n]^G$ st $I = \langle f_1, \_, f_r \rangle$; write $d_i = \deg f_i > 0$.

We <u>claim</u> that $f_1, -, f_r$ generate $k[\underline{x}_n]^G$ as a k-algebra. ETS

(*) Given $f \in k[\underline{x}_n]^G$ s.t. $\deg f = d \geq 0$ & f is <u>homogeneous</u>, we have $f \in k[f_1, -, f_r]$.

We prove (*) by <u>induction</u> on d. If $d=0$ then $f \in k$. So assume $d > 0$.

Then $f \in k[\underline{x}_n]^G_+ \subseteq I$, so $\exists\ h_1, -, h_r \in k[\underline{x}_n]$ st

(†) $f = h_1 f_1 + \cdots + h_r f_r$.

We may assume h_i is homog. st $\deg h_i = d - d_i < d$.

Apply R_G to both sides of (†):

$$\begin{array}{ccc} R_G(f) & = & R_G(h_1 f_1 + - + h_r f_r) \\ \| & & \| \\ f & & R_G(h_1) \cdot f_1 + - + R_G(h_r) \cdot f_r . \end{array}$$

But $R_G(h_i) \in k[\underline{x}_n]^G$ is homog of $\deg < d$,

so <u>by induction</u> $R_G(h_i) \in k[f_1, -, f_r]$.

So $f \in k[f_1, -, f_r]$. ↯