Math 140A

Fall 2020

Prof. Seward

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Class Information

Webpage: math.ucsd.edu/~bseward/140a_fall20/
Professor: Brandon Seward (he/him/his)
Call me Brandon
My Office Hours: Wed 12:00 - 1:00 & 2:00 - 4:00
TA: Srivatsa Srinivas (he/him/his)
TA's Office Hours: M 8-9 AM, Tu 6-7 PM, W 6-7 PM, Th 8-9 AM

Textbook: Principles of Mathematical Analysis (3^{rd} ed.) by W. Rudin
Q & A: Piazza
Turning in HW and Exams: Gradescope
Lecture Notes will be posted on course webpage
Lecture Videos will be posted on canvas

Grading: Letter grades assigned via a curve based on the max of two weighted averages:

1. 20% HW + 20% 1^{st} exam + 20% 2^{nd} exam + 40% final
2. 20% HW + 20% best midterm + 60% final

Homework: Due most Fridays at 9:00 pm. Lowest HW grade will be dropped. Some problems will be graded for correctness, others will be graded for completion.

Exams: Open book and open note. Help from online resources and other humans is prohibitted. Suspicion of cheating will lead to one-on-one Zoom meetings where you must solve similar problems.

1^{st} Midterm: Wed. Oct. 28
2^{nd} Midterm: Wed. Nov. 25
Final Exam: Tues. Dec. 15 11:30 AM - 2:30 PM

Course Summary

We will take for granted (and without proof) the basic properties of $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ (the set of natural numbers), $\mathbb{Z}$ (the set of integers), and $\mathbb{Q}$ (the set of rational numbers).

We will explore in detail the properties of $\mathbb{R}$ (the set of real numbers). With great care and precision we will define what the real numbers are. All of our prior knowledge and beliefs about $\mathbb{R}$ will be held in suspicion until we can find proofs of those properties based on our formal definition of $\mathbb{R}$.

The longterm goal is to provide the logical and theoretical justification for calculus (in 140B) and go beyond (in 140C).

This is a theoretical and philosophical course that requires a strong ability in reading, writing, and understanding proofs.

We will often focus on specific features of $\mathbb{R}$ and study those features in more abstract settings. CAUTION: Don't assume that we are always working with numbers or sets of numbers. More than 50% of the time we will not be!

Lecture 1 Oct. 2

Defn: Let S be a set. An order on S, denoted $<$, is a relation satisfying:

- (Trichotomy law) for all $x, y \in S$ exactly one of the statements "$x<y$", "$x=y$", "$y<x$" is true
- (Transitivity) for all $x, y, z \in S$ if $x<y$ and $y<z$ then $x<z$.

An ordered set is a set on which an order is defined.

Note: For convenience we write

- $y > x$ to mean $x < y$
- $x \leq y$ to mean $x<y$ or $x=y$

Defn: Let S be an ordered set and $E \subseteq S$. If there is $b \in S$ with $x \leq b$ for all $x \in E$ then we say E is bounded above and call b an upper bound to E.

Bounded below and lower bound are defined similarly.

Defn: Let S be an ordered set and $E \subseteq S$.
We call $\alpha \in S$ the <u>least upper bound</u> of E or the <u>supremum</u> of E, denoted $\alpha = \sup E$, if:

1. α is an upperbound to E
2. whenever $x \in S$ and $x < \alpha$, x is not an upperbound to E.

The <u>greatest lower bound</u> or <u>infimum</u> of E is defined similarly and denoted $\inf E$.

Ex: For $E = \{ \frac{1}{n} : n \in \mathbb{Z}_+ \} \subseteq \mathbb{Q}$

$$\sup E = 1, \quad \inf E = 0$$

Notice $\sup E \in E$ and $\inf E \notin E$.
In general, $\sup E$ and $\inf E$ may or may not be elements of E

Defn: An ordered set S has the <u>least upperbound (lub) property</u> if:

whenever $E \subseteq S$ is nonempty and bounded above, $\sup E$ exists.

Ex: $\mathbb{Q}$ does not have the lub property.

Recall $\sqrt{2} \notin \mathbb{Q}$. Set

$$A = \{ p \in \mathbb{Q} : p \leq 0 \text{ or } p^2 \leq 2 \}$$

$$B = \{ p \in \mathbb{Q} : p > 0 \text{ and } p^2 \geq 2 \}$$

Then $\mathbb{Q} = A \cup B$, B is the set of upperbounds to A and A is the set of lower bounds to B.

Next time

But A has no largest element and B has no smallest element, so $\sup A$ and $\inf B$ do not exist (when using $\mathbb{Q}$).

This implies that $\mathbb{Q}$ does not have lub property.

Lecture 2 Oct 5

HW1 Due Friday @ 9:00 PM

$A = \{p \in \mathbb{Q} : p \le 0 \text{ or } p^2 \le 2\}$
$B = \{p \in \mathbb{Q} : p > 0 \text{ and } p^2 \ge 2\}$

But A has no largest element and B has no smallest element,

This is not the focus of the conversation... but here is why:

For $p \in \mathbb{Q}$, $p > 0$ set

$$① \quad q = p - \frac{p^2-2}{p+2} \in \mathbb{Q}$$

Then $q = \frac{2p+2}{p+2}$ so $q^2 = 2 + \frac{2(p^2-2)}{(p+2)^2}$ ②

Suppose $p \in A$.

<u>Case 1:</u> $p \le 0$. Then $p < 1$ and $1 \in A$

<u>Case 2:</u> $p > 0$. Then $q > p$ (by ①) and $q \in A$ (by ②)

So p is not the largest element of A.

Suppose $p \in B$.

Then $q < p$ (by ①) and $q \in B$ (by ②)

So p is not the smallest element of B.

Thm 1.11 If S has the lub property then it has the "greatest lowerbound property":
if $E \subseteq S$ is nonempty and bounded below then $\inf E$ exists.

Pf: Let $E \subseteq S$ be nonempty and bounded below.
Let A be the set of all lower bounds to E.
Then A is nonempty and bounded above
(every $e \in E$ is an upper bound)
So $\alpha = \sup A$ exists by lub property.
We will check $\alpha = \inf E$

(Check α is lowerbound to E).
Consider any $e \in E$. By definition of A, we have $\forall a \in A$ $a \leq e$. Thus e is an upperbound to A. Since α is the least upperbound to A, we have $\alpha \leq e$. Thus α is a lowerbound to E

(Check anything bigger than α is not a lowerbound)
If x is a lowerbound to E, then by definition $x \in A$.
Therefore $x \leq \sup A = \alpha$

We conclude that $\inf E = \alpha$ exists. $\square$

Defn: A **field** is a set F with two binary operations $+$ and $\cdot$, called **addition** and **multiplication**, with the following properties:

Commutativity: $\forall a,b \in F$ $a+b=b+a$, $a\cdot b = b\cdot a$
Associativity: $\forall a,b,c \in F$ $(a+b)+c = a+(b+c)$, $(a\cdot b)\cdot c = a\cdot(b\cdot c)$
Identity: there are $0 \neq 1 \in F$ with $\forall a \in F$ $0+a=a$ and $1\cdot a = a$
Inverse: for every $a \in F$ there is an element $-a \in F$ with $a+(-a)=0$
for every $a \in F$ with $a \neq 0$ there is $\frac{1}{a} \in F$ with $a\cdot\left(\frac{1}{a}\right)=1$
Distributivity: $\forall a,b,c \in F$ $a\cdot(b+c) = (a\cdot b)+(a\cdot c) = a\cdot b + a\cdot c$

Ex: These are fields:

- $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$
- $\mathbb{Q}(t) = \left\{ \frac{p(t)}{q(t)} : p, q \text{ polynomials in } t \text{ with coefficients in } \mathbb{Q} \right\}$
- Set of conjugacy classes mod p for p a prime.

Note: We write: $x-y$, $\frac{x}{y}$, 2, $2x$, x^2, etc
for: $x+(-y)$, $x\cdot\left(\frac{1}{y}\right)$, $1+1$, $x+x$, $x\cdot x$, etc

Prop 1.14: For a field F and $x, y, z \in F$

1. $x+y = x+z \implies y = z$
2. $x+y = 0 \implies y = -x$
3. $x+y = x \implies y = 0$
4. $-(-(x)) = x$

Pf: 1. If $x+y = x+z$ then
$$y = 0+y = -x+x+y = -x+x+z = 0+z = z$$

2. Apply 1 with $z = -x$
3. Apply 1 with $z = 0$
4. Since $-x+x = x+(-x) = 0$, 2 implies $x = -(-x)$

Prop 1.15: Let F be a field. Then for all $x, y, z \in F$ with $x \neq 0$ we have:

A. $xy = xz \implies y = z$
B. $xy = 1 \implies y = \frac{1}{x}$
C. $xy = x \implies y = 1$
D. $\frac{1}{\left(\frac{1}{x}\right)} = x$

Pf: Basically identical to the previous proofs, just use multiplication in place of addition. $\square$

Lecture 3 Oct 7

HW1 due Friday 9:00 pm

Prop 1.16: For any field F and $x, y \in F$

1. $0 \cdot x = 0$
2. $x \neq 0$ and $y \neq 0 \Rightarrow x \cdot y \neq 0$
3. $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$
4. $(-x) \cdot (-y) = x \cdot y$

Pf: ① $0 \cdot x + 0 \cdot x = (0+0) \cdot x = 0 \cdot x$

So $0 \cdot x = 0$ by Prop. 1.14

② Since

$$0 \cdot \left(\frac{1}{x} \cdot \frac{1}{y}\right) = 0 \qquad (\text{by } ①)$$
$$(x \cdot y) \cdot \left(\frac{1}{x} \cdot \frac{1}{y}\right) = 1$$

we must have $x \cdot y \neq 0$

③ $(-x) \cdot y + x \cdot y = (-x + x) \cdot y = 0 \cdot y = 0$

so $(-x) \cdot y = -(x \cdot y)$ by Prop. 1.14

Similarly, $x \cdot (-y) = -(x \cdot y)$

④ $(-x) \cdot (-y) = -(x \cdot (-y)) = -(-(x \cdot y)) = x \cdot y$ by Prop. 1.14

$\square$

Defn: An **ordered field** is a field F with an ordering such that:

1. $\forall x, y, z \in F \quad y < z \Rightarrow x + y < x + z$
2. $\forall x, y \in F \quad (x > 0 \text{ and } y > 0) \Rightarrow x \cdot y > 0$

We call $x \in F$ **positive** if $x > 0$ and **negative** if $x < 0$.

Ex: $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{Q}(t)$ are ordered fields

(I'll make a piazza post for $\mathbb{Q}(t)$)

Prop 1.18: For an ordered field F and $x, y, z \in F$

1. $x > 0 \iff -x < 0$
2. $(x > 0 \text{ and } y < z) \implies x \cdot y < x \cdot z$
3. $(x < 0 \text{ and } y < z) \implies x \cdot y > x \cdot z$
4. $x \neq 0 \implies x^2 > 0 \quad (\text{so } 1 = 1^2 > 0)$
5. $0 < x < y \implies 0 < \frac{1}{y} < \frac{1}{x}$

Pf: ① $x > 0 \iff x + (-x) > 0 + (-x) \iff 0 > -x \iff -x < 0$

② Since $y < z$, $0 = y - y < z - y$ so $x \cdot (z - y) > 0$ and

$x \cdot z = x(z-y) + xy > 0 + x \cdot y = x \cdot y$

③ By ② & ①, $(-x) \cdot y < (-x) \cdot z$ so

$x \cdot z = (-x) \cdot y + x \cdot y + x \cdot z < (-x) \cdot z + x \cdot y + x \cdot z = x \cdot y$

④ If $x > 0$ then $x^2 > 0$ by defn. of ordered field.
If $x < 0$ then $-x > 0$ so $(-x)^2 > 0$. But $(-x)^2 = x^2$ by Prop. 1.16

⑤ Assume $0 < x < y$.
If $z \leq 0$ then $yz \leq 0$ by ②
Since $y \cdot \frac{1}{y} = 1 > 0$ we must have $\frac{1}{y} > 0$.
Similarly $\frac{1}{x} > 0$.
Finally multiplying $x < y$ by positive $\frac{1}{x}, \frac{1}{y}$ we get

$$\frac{1}{y} = x \cdot \frac{1}{x} \cdot \frac{1}{y} < y \cdot \frac{1}{x} \cdot \frac{1}{y} = \frac{1}{x} \qquad \square$$

Thm 1.19 There exists a unique ordered field having the least upper bound property. Moreover, this field contains $\mathbb{Q}$.
We denote this field $\mathbb{R}$ and call its elements real numbers.

We can't prove this in class.
Existence is proven via a construction in the appendix to Ch. 1
I'll post in Piazza about uniqueness and containment of $\mathbb{Q}$.

Thm 1.20: (A) If $x, y \in \mathbb{R}$ and $x > 0$ then $\exists n \in \mathbb{N}$ $n \cdot x > y$
(B) If $x, y \in \mathbb{R}$ and $x < y$ then $\exists p \in \mathbb{Q}$ with $x < p < y$.

Pf: (A) Towards a contra, suppose $\forall n \in \mathbb{N}$ $n \cdot x \leq y$.
Set $A = \{ n \cdot x : n \in \mathbb{N} \}$.
A is bounded above by y so $\alpha = \sup A$ exists.
Since $x > 0$, $\alpha - x < \alpha$
So $\alpha - x$ is not upperbound to A, meaning there is $n \in \mathbb{N}$ with $n \cdot x > \alpha - x$.
Then $(n+1) \cdot x > \alpha$, contradicting $\alpha = \sup A$ and $(n+1) \cdot x \in A$.

To be continued next class

Lecture 4 Oct 9

HW 1 due today at 9:00 pm

- Continuing the proof from last class

(B) Since $x<y$ we have $y-x>0$.
So by (A) there is $n \geq 1$ with $n \cdot (y-x) > 1$.
Applying (A) twice more, we get integers $m_1, m_2 \geq 1$
with $m_1 > n \cdot x$
and $m_2 > -n \cdot x$
So $-m_2 < nx < m_1$.
So the finite set $\{-m_2, -m_2+1, \dots, m_1\}$
must contain a least m with $n \cdot x < m$.
Since m is least, $m-1 \leq n \cdot x < m$.
Therefore $n \cdot x < m \leq n \cdot x + 1 < n \cdot y$
and $x < \frac{m}{n} < y$. $\square$

Note: (A) is known as the archimedean property
(B) says that $\mathbb{Q}$ is dense in $\mathbb{R}$
(we'll define dense next week)

Thm 1.21: If $x \in \mathbb{R}$ is positive and $n \in \mathbb{Z}_+$
then there is a unique real $y > 0$ with $y^n = x$.
This number y is denoted $\sqrt[n]{x}$ or $x^{1/n}$

Pf:

Claim: If $0<y_1<y_2$ then $y_1^n<y_2^n$.

Pf of Claim: Since $\frac{y_2}{y_1}>1$, we have

$$\frac{y_2^n}{y_1^n}=\left(\frac{y_2}{y_1}\right)^n>1$$

hence $y_1^n<y_2^n$ $\square$(Claim)

So if y exists, it must be unique.

Set $E=\{t\in\mathbb{R} : t>0,\ t^n<x\}$.

(Check $E\neq\emptyset$). If $t=\frac{x}{x+1}$ then $t<1$

so $t^n<t<x$ and hence $t\in E$.

(Check $1+x$ is upperbound to E).

$t>1+x \Rightarrow t^n>t>x \Rightarrow t\notin E$

By lub property, $y=\sup E$ exists.

We will check $y^n=x$.

Recall the identity $b^n-a^n=(b-a)(b^{n-1}+a\cdot b^{n-2}+\cdots+a^{n-2}\cdot b+a^{n-1})$.

It follows

$$b^n-a^n<(b-a)\,n\,b^{n-1} \quad \text{when } 0<a<b$$

Towards a contra., suppose $y^n < x$.
Choose h with
$$0 < h < \min\left(1, \frac{x - y^n}{n(y+1)^{n-1}}\right)$$

Then $(y+h)^n - y^n < h \cdot n \cdot (y+h)^{n-1} \leq h \cdot n \cdot (y+1)^{n-1} < x - y^n$.
So $y+h \in E$ and $y+h > y$, contradicting y being upper bound to E.

Towards a contra., suppose $y^n > x$.
Set $k = \frac{y^n - x}{n \cdot y^{n-1}}$. Then $0 < k < y$.

If $t \geq y-k$ then $y^n - t^n \leq y^n - (y-k)^n < k \cdot n \cdot y^{n-1} = y^n - x$
So $t^n > x$ and $t \notin E$. Thus $y-k$ is an upper bound to E and $y-k < y$, contradicting $y = \sup E$. $\square$

Note: It is possible to define decimal representations of real numbers. See the book.

Defn: The *extended real number system* is the set $\mathbb{R} \cup \{-\infty, +\infty\}$ where for all $x \in \mathbb{R}$

- $-\infty < x < \infty$
- $x + \infty = +\infty$, $x - \infty = -\infty$, $\frac{x}{+\infty} = 0$, $\frac{x}{-\infty} = 0$
- $x > 0 \Rightarrow x \cdot (+\infty) = +\infty$, $x(-\infty) = -\infty$
- $x < 0 \Rightarrow x \cdot (+\infty) = -\infty$, $x(-\infty) = +\infty$

All other operations are left undefined.
This is not a field!

To distinguish $x \in \mathbb{R}$ from $-\infty, +\infty$, we call x *finite*.

Defn: The set of *complex numbers* is
$$\mathbb{C} = \{(a,b) : a, b \in \mathbb{R}\}.$$
Note $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$.
For $x, y \in \mathbb{C}$, say $x = (a,b)$ and $y = (c,d)$, we define
$$x + y = (a+c,\ b+d)$$
$$x \cdot y = (ac - bd,\ ad + bc)$$

Thm: $\mathbb{C}$ is a field with $(0,0)$ and $(1,0)$ playing the roles of 0 and 1

Pf: We check existence of inverses.
Other properties are checked by computation.
Let $x = (a,b) \in \mathbb{C}$.
Write $-x = (-a,-b)$. Then
$$x + (-x) = (a,b) + (-a,-b) = (0,0)$$
Now assume $x \neq 0$. Then $(a,b) \neq (0,0)$
so $a \neq 0$ or $b \neq 0$. Thus $a^2 + b^2 > 0$.
Write $\frac{1}{x} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$. Then
$$x \cdot \frac{1}{x} = (a,b) \cdot \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1,0) = 1 \quad \square$$

Lecture 5 Oct 12

HW 2 due Friday

Please carefully read my email about the first exam and respond promptly if needed

Thm: For all $a,b \in \mathbb{R}$ $(a,0)+(b,0)=(a+b,0)$ and $(a,0)\cdot(b,0)=(a\cdot b,0)$

Pf: Easy to check $\square$

This means we can identify $a \in \mathbb{R}$ with $(a,0)$ and this identification preserves addition, and multiplication. So we can view $\mathbb{R}$ as a subfield of $\mathbb{C}$

Defn: $i=(0,1) \in \mathbb{C}$

Thm: $i^2=-1$

Pf: $i^2=(0,1)^2=(0,1)\cdot(0,1)=(-1,0)$ $\square$

Thm: If $a,b \in \mathbb{R}$ then $(a,b)=a+bi$

Pf: $a+bi=(a,0)+(b,0)\cdot(0,1)=(a,0)+(0,b)=(a,b)$

$\square$

Defn: For $z = a+bi \in \mathbb{C}$ we call
a the real part of z and
b the imaginary part of z
and write $a = \mathrm{Re}(z)$, $b = \mathrm{Im}(z)$.
We call $\bar{z} = a - bi$ the complex conjugate of z.

Thm: If $z, w \in \mathbb{C}$ then
(A) $\overline{z+w} = \bar{z} + \bar{w}$
(B) $\overline{zw} = \bar{z} \cdot \bar{w}$
(C) $z + \bar{z} = 2\,\mathrm{Re}(z)$, $z - \bar{z} = 2\,\mathrm{Im}(z)\,i$
(D) $z\bar{z} \in \mathbb{R}$ and $z\bar{z} > 0$ when $z \neq 0$

Pf: (A), (B) (C) are easy to check by computation.
(D) holds since $z = a+bi \Rightarrow z\bar{z} = a^2 + b^2$ $\square$

Defn: The absolute value of $z \in \mathbb{C}$ is
defined $|z| = (z\bar{z})^{1/2}$

Note: If $x \in \mathbb{R}$ then $\bar{x} = x$ so $|x| = \sqrt{x^2}$
meaning $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

Thm: If $z, w \in \mathbb{C}$ then
1. $|z| > 0$ unless $z = 0$, $|0| = 0$
2. $|\bar{z}| = |z|$
3. $|zw| = |z| \cdot |w|$
4. $|\text{Re}(z)| \le |z|$
5. $|z+w| \le |z| + |w|$

Pf: 1 and 2 easily checked by computation.

3. $|zw| = (zw \cdot \overline{zw})^{1/2} = (zw\bar{z}\bar{w})^{1/2}$
$$= (z\bar{z}w\bar{w})^{1/2} = (z\bar{z})^{1/2}(w\bar{w})^{1/2} = |z||w|$$

4. Say $z = a+bi$. Then
$$a^2 \le a^2 + b^2 \text{ so}$$
$$|\text{Re}(z)| = |a| = \sqrt{a^2} \le \sqrt{a^2+b^2} = \sqrt{z \cdot \bar{z}} = |z|$$

5. Note $\overline{\bar{z}w} = z \cdot \bar{w}$ so $\bar{z}w + z\bar{w} = 2\text{Re}(zw)$ by ©

Therefore $|z+w|^2 = (z+w)(\bar{z}+\bar{w})$
$$= z\bar{z} + w\bar{z} + \bar{w}z + w\bar{w}$$
$$= |z|^2 + 2\text{Re}(\bar{w}z) + |w|^2$$
$$\overset{4}{\le} |z|^2 + 2|z\bar{w}| + |w|^2$$
$$= |z|^2 + 2|z||w| + |w|^2$$
$$= (|z| + |w|)^2 \qquad \square$$

Thm (Cauchy-Schwarz Inequality)

If $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{C}$ then

$$\left|\sum_{k=1}^{n} a_k \overline{b_k}\right|^2 \leq \sum_{k=1}^{n} |a_k|^2 \cdot \sum_{k=1}^{n} |b_k|^2$$

Pf: Next class

Defn: For $k \in \mathbb{Z}_+$ we let $\mathbb{R}^k$ be the set of all k-tuples

$$\vec{x} = (x_1, x_2, \dots, x_k), \quad x_i \in \mathbb{R}$$

We call $\vec{x}$ a point or a vector

$\vec{0} = (0, 0, \dots, 0)$ is the origin

$\mathbb{R}^k$ is an example of a vector space, with operations

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \dots, x_k + y_k), \quad \vec{x}, \vec{y} \in \mathbb{R}^k$$

$$\alpha \cdot \vec{x} = (\alpha x_1, \alpha x_2, \dots, \alpha x_k) \quad \vec{x} \in \mathbb{R}^k, \alpha \in \mathbb{R}$$

The inner product (or dot product) is

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{k} x_i y_i$$

The norm of $\vec{x} \in \mathbb{R}^k$ is

$$|\vec{x}| = (\vec{x} \cdot \vec{x})^{1/2} = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2}$$

$\mathbb{R}^k$ is called k-dimensional Euclidean space

Lecture 6 Oct 14

HW 2 due Friday

First exam will be offered at two times:

- Class time (11-11:50 AM Wed. Oct 28 San Diego local time
- 12 hours prior (11-11:50 PM Tues. Oct. 27 San Diego local time

If you can't take the exam at either of these times you must email me by Saturday

Thm (Cauchy-Schwarz Inequality):
If $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n \in \mathbb{C}$ then

$$\left|\sum_{k=1}^{n} a_k \overline{b_k}\right|^2 \le \sum_{k=1}^{n} |a_k|^2 \cdot \sum_{k=1}^{n} |b_k|^2$$

Note: If $(a_1, \cdots, a_n) = \vec{a} \in \mathbb{R}^n$, $(b_1, \cdots, b_n) = \vec{b} \in \mathbb{R}^n$

this says $|\vec{a} \cdot \vec{b}|^2 \le |\vec{a}|^2 |\vec{b}|^2$.
From geometry/intuition, we expect equality to hold precisely

$$B\vec{a} = C\vec{b},$$

where $B = \sum_{k=1}^{n} |b_k|^2$, $C = \sum_{k=1}^{n} a_k \overline{b_k}$

Pf: Define B, C as above. Set $A = \sum_{k=1}^{n} |a_k|^2$.
If $B = 0$ then $b_1 = b_2 = \cdots = b_n = 0$ and conclusion is trivial. So assume $B > 0$.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$0 \le \sum_{k=1}^{n} |Ba_k - Cb_k|^2$$
$$= \sum_{k=1}^{n} (Ba_k - Cb_k)(B\bar{a}_k - \bar{C}\,\bar{b}_k)$$
$$= B^2 \sum_{k=1}^{n} |a_k|^2 - B\bar{C}\sum_{k=1}^{n} a_k \bar{b}_k - BC\sum_{k=1}^{n} \bar{a}_k b_k + |C|^2 \sum_{k=1}^{n} |b_k|^2$$
$$= B^2 A - B|C|^2 - B|C|^2 + B|C|^2$$
$$= B^2 A - B|C|^2$$
$$= B(BA - |C|^2)$$

Since $B > 0$, we get $BA - |C|^2 \ge 0$. $\square$

Note: $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$
$\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$

Thm: If $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$ then

A $|\vec{x}| \ge 0$ and $(|\vec{x}| = 0 \Leftrightarrow \vec{x} = \vec{0})$

B $|\alpha\cdot\vec{x}| = |\alpha|\,|\vec{x}|$

C $|\vec{x}\cdot\vec{y}| \le |\vec{x}|\,|\vec{y}|$

D $|\vec{x}+\vec{y}| \le |\vec{x}| + |\vec{y}|$

E $|\vec{x}-\vec{z}| \le |\vec{x}-\vec{y}| + |\vec{y}-\vec{z}|$

Pf: (A) and (B) are easy to check.
(C) follows from Cauchy-Schwarz
(D) $|\vec{x}+\vec{y}|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$

$$= |\vec{x}|^2 + \vec{y}\cdot\vec{x} + \vec{x}\cdot\vec{y} + |\vec{y}|^3$$
$$= |\vec{x}|^2 + 2\vec{x}\cdot\vec{y} + |\vec{y}|^2$$
$$\leq |\vec{x}|^2 + 2|\vec{x}||\vec{y}| + |\vec{y}|^2 \quad \text{by (C)}$$
$$= (|\vec{x}|+|\vec{y}|)^2$$

(E) follows from (D)
using $\vec{x} = \vec{x}-\vec{y}$, $\vec{y} = \vec{y}-\vec{z}$ $\square$

Defn: Let $f: X \to Y$.
The *image* of $A \subseteq X$ is $f(A) = \{f(a) : a \in A\}$
The *preimage* of $B \subseteq Y$ is $f^{-1}(B) = \{x \in X : f(x) \in B\}$
For $y \in Y$ we write $f^{-1}(y)$ for $f^{-1}(\{y\})$

Defn: Two sets X, Y *have equal cardinality*, denoted $|X| = |Y|$, if $\exists$ bijection $f: X \to Y$.

- X is *finite* if $X = \emptyset$ or $\exists n \in \mathbb{Z}_+$ $|X| = |\{1, 2, \dots, n\}|$.
  Otherwise X is *infinite*
- X is *countable* if it is finite or $|X| = |\mathbb{N}|$.
  X is *uncountable* otherwise.

Defn: A *sequence* is a function f with domain $\mathbb{N}$ or $\mathbb{Z}_+$. When $f(n) = x_n$ for each n, we write $(x_n)_{n \in \mathbb{N}}$ (or $(x_n)_{n \in \mathbb{Z}_+}$) to denote f.

Thm: If X is cntbl and $A \subseteq X$ then A is cntbl.

Pf: This is obvious if A is finite. So assume A is infinite. Then X is infinite so $|X| = |\mathbb{N}|$. So we can list elements of X as

$$\{x_0, x_1, x_2, \dots\}$$

Let $n_0 \in \mathbb{N}$ be least with $x_{n_0} \in A$. Inductively, after choosing $n_0, \dots, n_{k-1}$ pick $n_k > n_{k-1}$ to be least with $x_{n_k} \in A$.

Now define $f : \mathbb{N} \to A$ by $f(k) = x_{n_k}$. Then f is a bijection. $\square$

Thm: If $(X_n)_{n \in \mathbb{N}}$ is a seq. of cntbl sets then $\bigcup_{n \in \mathbb{N}} X_n$ is cntbl.

Pf: For $n \in \mathbb{N}$, let $(x_{n,k})_{k \in \mathbb{N}}$ be a seq. in X_n that uses every element of X_n at least once.

$$\begin{array}{lllll}
X_0 & x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} \quad \dots \\
X_1 & x_{1,0} & x_{1,1} & x_{1,2} & \dots \\
X_2 & x_{2,0} & x_{2,1} & x_{2,2} & \dots \\
X_3 & x_{3,0} & \dots & & \\
\vdots & \vdots & & &
\end{array}$$

Let f be the seq

$$x_{0,0},\ x_{1,0},\ x_{0,1},\ x_{2,0},\ x_{1,1},\ x_{0,2},\ \dots$$

Then f is onto $\bigcup_{n\in\mathbb{N}} X_n$

Set $A = \{n \in \mathbb{N} : \forall k<n \; f(k) \neq f(n)\}$
Then A is cntbl by prior theorem and
$f: A \to \bigcup_{n\in\mathbb{N}} X_n$ is a bijection. $\square$

Thm: If X is cntbl then $X^n = \underbrace{X \times X \times \cdots \times X}_{n \text{ copies}}$ is cntbl

Pf: We use induction. $X^1 = X$ is cntbl.
If X^{n-1} is cntbl then each set
$\{x\} \times X^{n-1}$ is cntbl, so

$$X^n = \bigcup_{x \in X} \{x\} \times X^{n-1}$$

is cntbl by previous theorem. $\square$

Lecture 7 Oct 16

HW 2 due today

First exam will be offered at two times:

- Class time (11-11:50 AM Wed. Oct 28 San Diego local time
- 12 hours prior (11-11:50 PM Tues. Oct. 27 San Diego local time

If you can't take the exam at either of these times you must email me by Saturday

Cor: $\mathbb{Q}$ is cntbl

Pf: Note by prior theorems that every subset of $\mathbb{Z}^2$ is countable. Define $f : \mathbb{Q} \to \mathbb{Z}^2$ by setting $f(q) = (a, b)$ where $a, b \in \mathbb{Z}$ satisfy $b > 0$, $\frac{a}{b} = q$, and a, b coprime. Then f is a bijection with its image which is cntbl. $\square$

Thm: The set $\{0,1\}^{\mathbb{N}}$ of all functions $f : \mathbb{N} \to \{0,1\}$ is uncountable.

Pf: Let $F \subseteq \{0,1\}^{\mathbb{N}}$ be any cntbl infinite set. Say $F = \{f_0, f_1, f_2, \dots\}$, each $f_i : \mathbb{N} \to \{0,1\}$. Define $g : \mathbb{N} \to \{0,1\}$ by $g(n) = 1 - f_n(n)$. Then $\forall n \in \mathbb{N}$ $g \neq f_n$ since $g(n) \neq f_n(n)$. So $g \in \{0,1\}^{\mathbb{N}} \setminus F$. Thus $F \neq \{0,1\}^{\mathbb{N}}$. $\square$

Defn: A metric space is a pair (X, d) where X is a set and $d: X \times X \to \mathbb{R}$ satisfies:

1. $\forall p, q \in X$ $d(p,p) = 0$, if $p \neq q$ then $d(p,q) > 0$
2. $\forall p, q \in X$ $d(p,q) = d(q,p)$
3. (Triangle inequality) $\forall x, y, z \in X$ $d(x,z) \leq d(x,y) + d(y,z)$

The function d is called a metric.

Ex:
- $\mathbb{R}^k$ with $d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|$
- $\mathbb{C}$ with $d(z, w) = |z - w|$
- $\mathbb{R}^k$ with $d_p(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{k} (x_i - y_i)^p \right)^{1/p}$ $(p > 1)$
- $[0,1]^{[0,1]}$ with $d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$
- $\mathbb{R}^k$ with $d_\infty(\vec{x}, \vec{y}) = \max_{1 \leq i \leq k} |x_i - y_i|$

Defn: Let (X, d) be a metric space

- for $p \in X$, $r > 0$, the ball of radius r around x is
$$B_r(p) = \{q \in X : d(p,q) < r\}$$
- $p \in X$ is a limit point of $E \subseteq X$ if
$$\forall r > 0 \quad (B_r(p) \setminus \{p\}) \cap E \neq \emptyset$$
- set of limit points of $E \subseteq X$ is E'
- E is closed if $E' \subseteq E$
- E is perfect if $E' = E$
- E is dense in X if $E \cup E' = X$
- p is an interior point of E if $\exists r > 0$ $B_r(p) \subseteq E$
- set of interior points of E is denoted E°
- E is open if every point of E is an interior point of E
- $E^c = X \setminus E$ is the compliment of E
- E is bounded if $\exists M \in \mathbb{R}$ $\exists p \in X$ $E \subseteq B_M(p)$
- E is a neighborhood of p if E is open and $p \in E$.

Thm: $B_r(p)$ is always open.

Pf: If $q \in B_r(p)$ and $x \in B_{r-d(p,q)}(q)$ then
$$d(p,x) \leq d(p,q) + d(q,x)$$
$$< d(p,q) + r - d(p,q) = r$$
So $x \in B_r(p)$. Thus $B_{r-d(p,q)}(q) \subseteq B_r(p)$
so q is an interior point
and $B_r(p)$ is open. $\square$

Thm: If $p \in E'$ then for all $r > 0$
$$(B_r(p) \setminus \{p\}) \cap E \text{ is infinite.}$$

Pf: Let $r > 0$. Towards contradiction, suppose not.
Then
$$t = \min\{d(p,q) : q \in (B_r(p) \setminus \{p\}) \cap E\}$$
is positive. We have
$$\left(B_{\frac{1}{2}t}(p) \setminus \{p\}\right) \cap E = \emptyset$$
so $p \notin E'$, a contradiction. $\square$

Cor: E finite $\Rightarrow E' = \emptyset$

Thm: E is open $\iff E^c$ is closed

Pf:

Set $D_r = B_r(x) \setminus \{x\}$

$$\begin{aligned}
E^c \text{ closed} &\iff (E^c)' \subseteq E^c \\
&\iff (E^c)' \cap E = \emptyset \\
&\iff \forall x \in E \quad x \notin (E^c)' \\
&\iff \forall x \in E \ \exists r > 0 \quad D_r \cap E^c = \emptyset \\
&\iff \forall x \in E \ \exists r > 0 \ D_r \cap E^c = \emptyset \text{ and } x \notin E^c \\
&\iff \forall x \in E \ \exists r > 0 \quad B_r(x) \cap E^c = \emptyset \\
&\iff \forall x \in E \ \exists r > 0 \quad B_r(x) \subseteq E \\
&\iff \forall x \in E \quad x \in E^\circ \\
&\iff E \text{ is open.} \qquad \square
\end{aligned}$$

Lecture 8 Oct 19

HW 3 due Friday

First midterm next Wed at class time and 12 hours prior

Let (X,d) be metric space

Thm: Let A be any set (possibly uncountable)

① $(\forall \alpha \in A \quad U_\alpha \subseteq X$ is open$) \Rightarrow \bigcup_{\alpha\in A} U_\alpha$ is open

② $(\forall \alpha \in A \quad F_\alpha \subseteq X$ is closed$) \Rightarrow \bigcap_{\alpha \in A} F_\alpha$ is closed

③ If $U_1, \dots, U_n \subseteq X$ are open then $\bigcap_{i=1}^{n} U_i$ is open

④ If $F_1, \dots, F_n \subseteq X$ are closed then $\bigcup_{i=1}^{n} F_i$ is closed

Pf: ① If $x \in \bigcup_{\alpha \in A} U_\alpha$ then there is $\beta \in A$ with $x \in U_\beta$. Since U_β is open, there is $r > 0$ with $B_r(x) \subseteq U_\beta \subseteq \bigcup_{\alpha\in A} U_\alpha$. So x is an interior point of $\bigcup_{\alpha\in A} U_\alpha$ so $\bigcup_{\alpha \in A} U_\alpha$ is open.

② $\left(\bigcap_{\alpha\in A} F_\alpha\right)^c = \bigcup_{\alpha\in A} F_\alpha^c$ is open by ① so $\bigcap_{\alpha\in A} F_\alpha$ is closed.

③ Let $x \in \bigcap_{i=1}^{n} U_i$. Each U_i is open so we can pick $r_i > 0$ with $B_{r_i}(x) \subseteq U_i$. Set $r = \min(r_1, r_2, \dots, r_n)$. Then $r > 0$ and $B_r(x) \subseteq \bigcap_{i=1}^{n} U_i$. Thus $\bigcap_{i=1}^{n} U_i$ is open.

④ Take compliments like in ② $\square$

Note: Finiteness assumption is necessary in ③ and ④

Ex: $U_n = \left(\frac{-1}{n}, \frac{1}{n}\right) \subseteq \mathbb{R}$ is open but $\bigcap_{n\in\mathbb{Z}_+} U_n = \{0\}$ is not open.

Defn: The closure of $E \subseteq X$ is $\overline{E} = E \cup E'$

Thm: (A) $\overline{E}$ is closed

(B) $E = \overline{E} \Leftrightarrow E$ is closed

(C) $(F$ closed and $F \supseteq E) \Rightarrow F \supseteq \overline{E}$

Pf: (A) Let $x \in (\overline{E})'$. If $x \in E$ then we are done since $x \in \overline{E}$. So assume $x \notin E$. Let $r > 0$.
Since $x \in (\overline{E})'$ we have $(B_r(x) \setminus \{x\}) \cap \overline{E} \neq \emptyset$
Pick $y \in (B_r(x) \setminus \{x\}) \cap \overline{E}$.
Set $s = \min(r - d(x,y), d(x,y)) > 0$

Notice $B_s(y) \subseteq B_r(x) \setminus \{x\}$ (last class)

Since $y \in \overline{E}$ we must have $B_s(y) \cap E \neq \emptyset$.
So there is $p \in B_s(y) \cap E$, and we have

$$p \in (B_r(x) \setminus \{x\}) \cap E$$

Thus $(B_r(x) \setminus \{x\}) \cap E \neq \emptyset$ and hence $x \in E' \subseteq \overline{E}$

(B) $\overline{E}$ closed by (A) so $(E = \overline{E} \Rightarrow E$ closed$)$.
Conversely, E closed implies $E' \subseteq E$ hence $E = E \cup E' = \overline{E}$

(C) Whenever $F \supseteq E$ we have $F' \supseteq E'$.
So if F is closed and $F \supseteq E$ then
$F \supseteq E \cup F' \supseteq E \cup E' = \overline{E}$. $\square$

Thm: If $E \subseteq \mathbb{R}$ is nonempty and bounded above then $\sup E \in \overline{E}$. Similarly when $\inf E$, $\inf E \in \overline{E}$.

Pf: We prove the first statement. Second is similar. Set $y = \sup E$. If $y \in E$ then $y \in \overline{E}$ and we are done. So assume $y \notin E$. Let $r > 0$. Since $y = \sup E$ and $y - r < y$, by definition there must be $x \in E$ with $y - r < x$. Since $y = \sup E$ and $y \notin E$ we must have $x < y$. So $x \in (B_r(y) \setminus \{y\}) \cap E$. We conclude $y \in E' \subseteq \overline{E}$. $\square$

Note: If (X, d) is a metric space and $Y \subseteq X$ then (Y, d_Y) is a metric space where d_Y is the restriction of d to $Y \times Y$: for $y_1, y_2 \in Y$ $d_Y(y_1, y_2) = d(y_1, y_2)$.

Defn: If $E \subseteq Y \subseteq X$ we say E is <u>open relative to Y</u> if E is open in (Y, d_Y)

(equivalently, E is open rel. to Y if
$$\forall p \in E \ \exists r > 0 \ \ B_r(p) \cap Y \subseteq E \ )$$

<u>Closed relative to Y</u> is defined similarly.

Thm: Let $E \subseteq Y \subseteq X$. Then E is open rel. to Y if and only if there is open $U \subseteq X$ with

$$E = U \cap Y.$$

Pf: $(\Rightarrow)$ Assume E is open rel. to Y.
For each $p \in E$ pick $r_p > 0$ with $B_{r_p}(p) \cap Y \subseteq E$.

Set $U = \bigcup_{p \in E} B_{r_p}(p)$. Then U is open and $U \cap Y \subseteq E$. For every $p \in E$,

$p \in B_{r_p}(p) \subseteq U \cap Y$ so $E \subseteq U \cap Y$ and $E = U \cap Y$.

$(\Leftarrow)$ Assume there is open $U \subseteq X$ with $E = U \cap Y$.
Let $p \in E$. Then $p \in U$ and U open so there is $r > 0$ with $B_r(p) \subseteq U$. Hence

$$B_r(p) \cap Y \subseteq U \cap Y = E.$$

So E is open rel. to Y. $\square$

Lecture 9 Oct 21

HW 3 due Friday

First midterm next Wed at class time and 12 hours prior

Defn: Let (X,d) be a metric space. An open cover of $E \subseteq X$ is a collection $\{U_\alpha : \alpha \in A\}$ of open sets $U_\alpha \subseteq X$ with $E \subseteq \bigcup_{\alpha \in A} U_\alpha$

Defn: $K \subseteq X$ is compact if every open cover $\{U_\alpha : \alpha \in A\}$ of K contains a finite subcover, meaning there are $\alpha_1, \alpha_2, \dots, \alpha_n$ with $K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.
In other words, K is compact if the following statement is true:

for every collection $\{U_\alpha : \alpha \in A\}$ with each U_α open

$$\left(K \subseteq \bigcup_{\alpha \in A} U_\alpha \Rightarrow \exists n \ \exists \alpha_1, \alpha_2, \dots, \alpha_n \in A \ \ K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \right)$$

Note: Finite sets are compact.

Thm Assume $K \subseteq Y \subseteq X$. Then K is compact rel. to Y if and only if K is compact rel. to X
(Compactness is an intrinsic property)

Pf: ($\Rightarrow$) Assume K is compact rel. to Y.
Suppose $K \subseteq \bigcup_{\alpha \in A} U_\alpha$, each U_α open in X.
Then $U_\alpha \cap Y$ is open rel. to Y and since $K \subseteq Y$
$K \subseteq Y \cap \left(\bigcup_{\alpha \in A} U_\alpha \right) = \bigcup_{\alpha \in A} (U_\alpha \cap Y)$
So there are $\alpha_1, \dots, \alpha_n \in A$ with
$$K \subseteq \bigcup_{i=1}^{n} (U_{\alpha_i} \cap Y) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$
($\Leftarrow$) Assume K is compact rel. to X.
Suppose $K \subseteq \bigcup_{\alpha \in A} V_\alpha$, each V_α open rel. to Y.
By theorem from last class, there are open sets in X, $U_\alpha \subseteq X$, with $V_\alpha = U_\alpha \cap Y$.
Then $K \subseteq \bigcup_{\alpha \in A} U_\alpha$ so there are $\alpha_1, \dots, \alpha_n \in A$ with $K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.
Since $K \subseteq Y$,
$$K \subseteq Y \cap \left(\bigcup_{i=1}^{n} U_{\alpha_i} \right) = \bigcup_{i=1}^{n} (U_{\alpha_i} \cap Y) = \bigcup_{i=1}^{n} V_{\alpha_i} \qquad \square$$

Thm: Compact sets are closed.

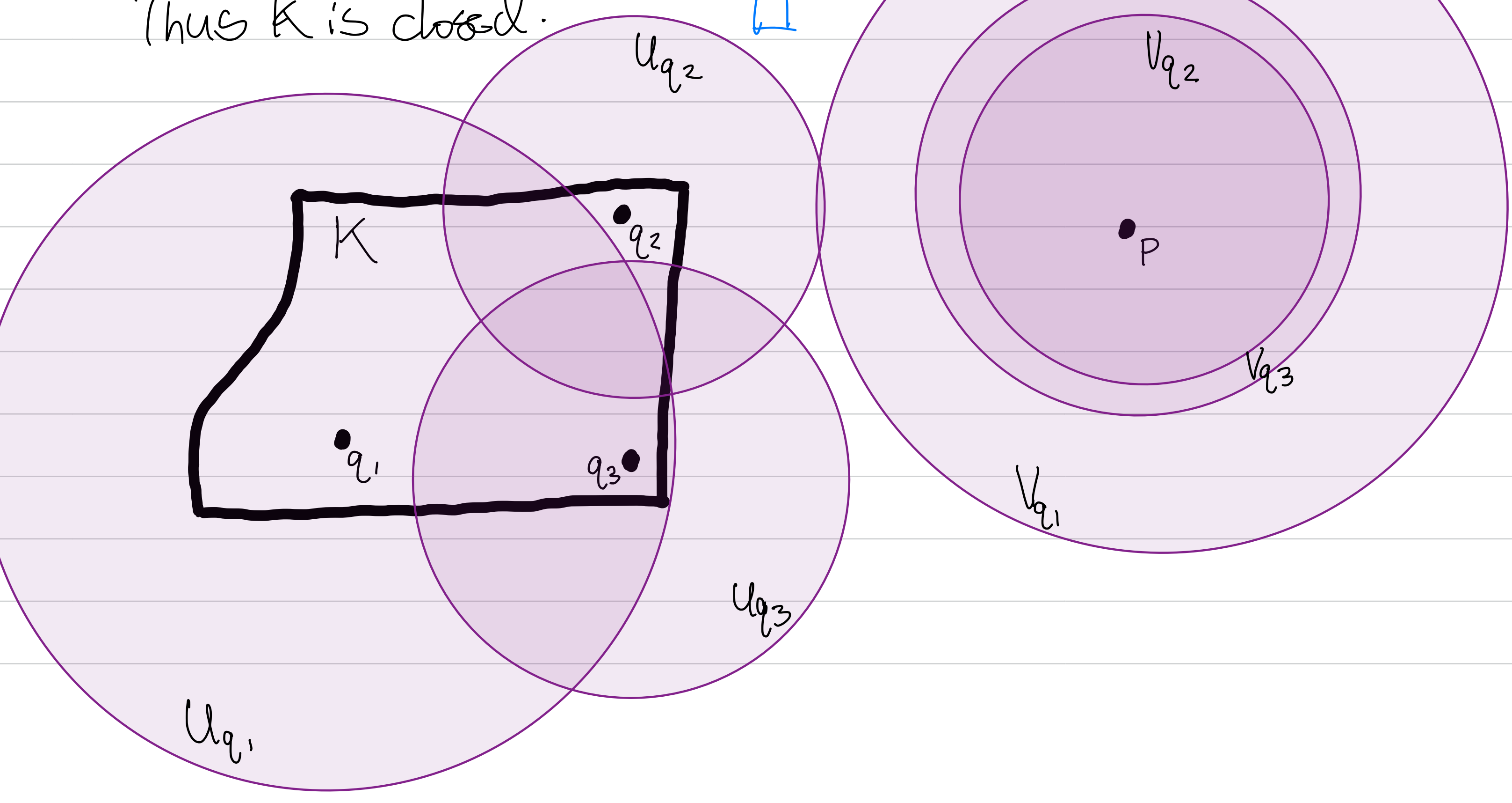
Pf: Let (X,d) be metric space, let $K \subseteq X$ be compact.
We will show $X \setminus K$ is open. So pick $p \in X \setminus K$.
For each $q \in K$ set

$$U_q = B_{\frac{1}{3}d(p,q)}(q), \quad V_q = B_{\frac{1}{3}d(p,q)}(p)$$

We have $K \subseteq \bigcup_{q \in K} U_q$, so by compactness there are $q_1, q_2, \dots, q_n \in K$ with $K \subseteq \bigcup_{i=1}^{n} U_{q_i}$

Then $\bigcap_{i=1}^{n} V_{q_i} = B_{\frac{1}{3}\min\{d(p,q_1), \dots, d(p,q_n)\}}(p)$

is disjoint with $\bigcup_{i=1}^{n} U_{q_i} \supseteq K$
hence a ball around p
is contained in $X \setminus K$.
Thus K is closed. $\square$



Thm: K compact and $F \subseteq K$ is closed then F is compact as well.

Pf: Say $\{U_\alpha : \alpha \in A\}$ is open cover of F. F^c is open so $\{F^c\} \cup \{U_\alpha : \alpha \in A\}$ is an open cover of K. So there are $\alpha_1, \dots, \alpha_n \in A$ with

$$F \subseteq K \subseteq F^c \cup \bigcup_{i=1}^n U_{\alpha_i} \text{ so } F \subseteq \bigcup_{i=1}^n U_{\alpha_i} \quad \Box$$

Cor: K cmpct, F closed $\Rightarrow K \cap F$ is compact.

Thm (Finite Intersection Property):
If $K_\alpha \subseteq X$ is cmpct for every $\alpha \in A$ and if the intersection of every finite collection from $\{K_\alpha : \alpha \in A\}$ is nonempty, then

$$\bigcap_{\alpha \in A} K_\alpha \neq \emptyset.$$

Pf: Towards contra, assume $\bigcap_{\alpha \in A} K_\alpha = \emptyset$. Fix any $K \in \{K_\alpha : \alpha \in A\}$. Then

$$K \subseteq X = X \setminus \emptyset = X \setminus \bigcap_{\alpha \in A} K_\alpha = \bigcup_{\alpha \in A} (X \setminus K_\alpha)$$

and each $X \setminus K_\alpha$ is open. So there are $\alpha_1, \dots, \alpha_n \in A$ with $K \subseteq \bigcup_{i=1}^n (X \setminus K_{\alpha_i})$. Then

$$K \cap \bigcap_{i=1}^n K_{\alpha_i} \subseteq \left(\bigcup_{i=1}^n (X \setminus K_{\alpha_i})\right) \cap \left(\bigcap_{i=1}^n K_{\alpha_i}\right) = \emptyset,$$

contradiction. $\Box$

Cor: If $K_n \neq \emptyset$ compact and $K_{n+1} \subseteq K_n$ for all n then $\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$.

Thm: K compact and $E \subseteq K$ is infinite then $E' \cap K \neq \emptyset$

Note: K is closed so $E' \subseteq K' \subseteq K$

Proof next class ...

Lecture 10 Oct 23

HW 3 due today

First Midterm Wednesday at class time and 12 hours prior

Thm 2.37: If K is compact and $E \subseteq K$ is infinite then $E' \cap K \neq \emptyset$.

Pf: Towards a contra., assume $E' \cap K = \emptyset$. This means for each $q \in K$ there is $r_q > 0$ with $(B_{r_q}(q) \setminus \{q\}) \cap E = \emptyset$, meaning $U_q = B_{r_q}(q)$ satisfies $U_q \cap E \subseteq \{q\}$.

Then $K \subseteq \bigcup_{q \in K} U_q$ so by compactness there are $q_1, \ldots, q_n \in K$ with $K \subseteq \bigcup_{i=1}^{n} U_{q_i}$. Then

$$|E| = |E \cap K| \leq \left|E \cap \left(\bigcup_{i=1}^{n} U_{q_i}\right)\right| = \left|\bigcup_{i=1}^{n} (E \cap U_{q_i})\right| \leq |\{q_1, \ldots, q_n\}| = n$$

So E is finite, contradiction. $\square$

Thm 2.38: If $I_n = [a_n, b_n]$, $a_n \leq b_n$, and $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$ then $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

Pf: For all n, m $\quad a_n \leq a_{n+m} \leq b_{n+m} \leq b_n$.
So b_m is upperbound to $\{a_n : n \in \mathbb{N}\}$ for all m. So $\alpha = \sup\{a_n : n \in \mathbb{N}\}$ exists and $a_m \leq \alpha \leq b_m$ for all m. Thus $\alpha \in \bigcap_{n \in \mathbb{N}} I_n$. $\square$

Thm 2.39: Suppose $C_n = [a_{n_1}, b_{n_1}] \times [a_{n_2}, b_{n_2}] \times \cdots \times [a_{n_k}, b_{n_k}] \subseteq \mathbb{R}^k$ and $C_n \neq \emptyset$ and $C_{n+1} \subseteq C_n$ for all n. Then $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$.

Pf: $\bigcap_{n\in\mathbb{N}} C_n = \left(\bigcap_{n\in\mathbb{N}} [a_{n_1}, b_{n_1}]\right) \times \cdots \times \left(\bigcap_{n\in\mathbb{N}} [a_{n_k}, b_{n_k}]\right) \neq \emptyset$ $\square$

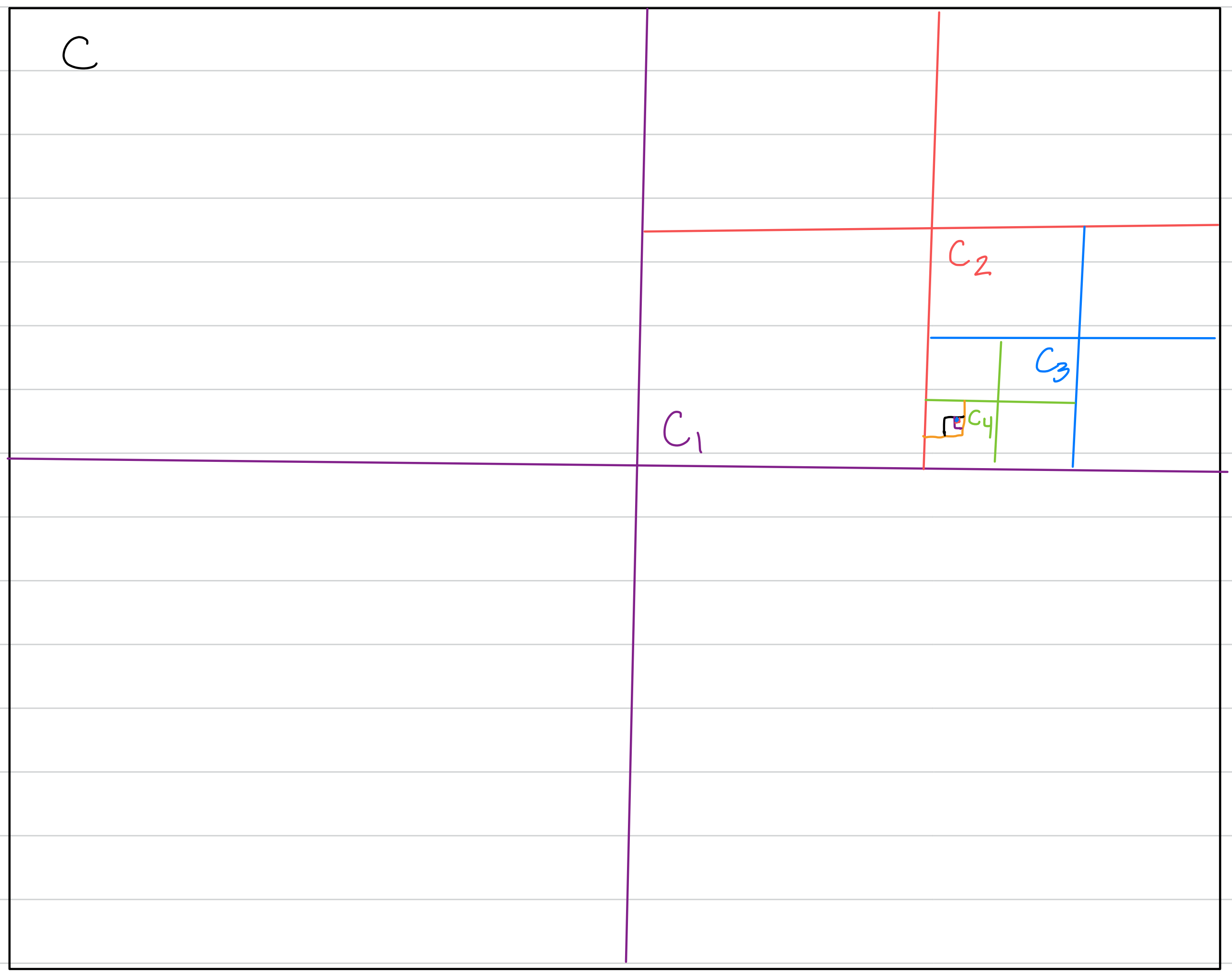
Thm 2.40: $C = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$ is compact.

Pf: Set $\delta = \sqrt{\sum_{i=1}^{k} |b_i - a_i|^2}$ (length of longest diagonal).
Towards contra., suppose $\{U_\alpha : \alpha \in A\}$ is an open cover of C having no finite subcover of C.
Cut at the midpoint of each side of C to divide C into 2^k many rectangles.
Some piece, call it C_1, does not admit a finite subcover. Proceed inductively repeating this process, building $(C_n)_{n\in\mathbb{Z}_+}$ with

1. $C \supseteq C_1 \supseteq C_2 \supseteq \cdots$
2. C_n does not admit a finite subcover from $\{U_\alpha : \alpha \in A\}$
3. $\forall \vec{x}, \vec{y} \in C_n \quad |\vec{x} - \vec{y}| \leq 2^{-n} \cdot \delta$

By prior theorem, $\bigcap_{n\in\mathbb{Z}_+} C_n \neq \emptyset$. Pick $\vec{z} \in \bigcap_{n\in\mathbb{Z}_+} C_n$.

Pick $\alpha \in A$ with $\vec{z} \in U_\alpha$. Since U_α open, there is $r > 0$ with $B_r(\vec{z}) \subseteq U_\alpha$. Pick n with $2^{-n} \cdot \delta < r$ then (3) implies $C_n \subseteq U_\alpha$ since $\vec{z} \in C_n$.
So C_n admits a finite subcover, contradicting (2) $\square$


C
C_1
C_2
C_3
C_4

Thm 2.41: For $E \subseteq \mathbb{R}^k$ the following are equivalent.

1. E is closed and bounded
2. E is compact
3. Every infinite subset of E has a limit point in E.

Pf: ($1 \Rightarrow 2$) E bounded means $E \subseteq C$, for some $C = [a_1, b_1] \times \cdots \times [a_k, b_k]$. C is compact and $E \subseteq C$ is closed so E is compact.

($2 \Rightarrow 3$) This is Theorem 2.37

($3 \Rightarrow 1$) Towards contra, suppose E is not bounded. Then for every $n \in \mathbb{N}$ we can find $\vec{x}_n \in E$ with $|\vec{x}_n| > n$. Set $S = \{\vec{x}_n : n \in \mathbb{N}\}$.

Claim: $S' = \emptyset$. Let $\vec{p} \in \mathbb{R}^k$. Then

$$\vec{x}_n \in B_1(\vec{p}) \Rightarrow |\vec{x}_n| \leq |\vec{p}| + 1 \Rightarrow n \leq |\vec{p}| + 1.$$

So $B_1(\vec{p}) \cap S$ is finite hence $\vec{p} \notin S'$.

Thus $S' = \emptyset$. This contradicts 3

Let $\vec{p} \in E'$. For each $n \in \mathbb{Z}_+$ pick $\vec{x}_n \in E$ with $0 < |\vec{x}_n - \vec{p}| < \frac{1}{n}$. Set $S = \{\vec{x}_n : n \in \mathbb{Z}_+\}$.

Claim: $S' = \{\vec{p}\}$. Clearly $\vec{p} \in S'$.

Consider $\vec{p} \neq \vec{q} \in \mathbb{R}^k$. Pick $N \in \mathbb{N}$ with $|\vec{q} - \vec{p}| > \frac{2}{N}$. If $\vec{x}_n \in B_{\frac{1}{N}}(\vec{q})$ then

$$|\vec{x}_n - \vec{p}| \geq |\vec{q} - \vec{p}| - |\vec{x}_n - \vec{q}| > \frac{2}{N} - \frac{1}{N} = \frac{1}{N}$$

and thus $n \leq N$. So $B_{\frac{1}{N}}(\vec{q}) \cap S$ is finite so $\vec{q} \notin S'$. Thus $S' = \{\vec{p}\}$.

3 implies $\vec{p} \in E$. Thus E closed. □

Note: Equivalence of ① and ② is known as the Heine-Borel Theorem.

Thm 2.42 (Bolzano-Weierstrauss): Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Pf: Take the closure and apply ① ⇒ ③ of previous theorem. $\square$

Lecture 11 Oct 26

First Midterm on Wednesday - see course webpage for detailed instructions
My office hours this week: M 6:30-8:00 PM, Tu 12:00-1:30 PM, Th 1:00-2:00 PM
HW 4 due Friday

Recall: $p \in E' \Rightarrow \forall r > 0 \quad (B_r(p) \setminus \{p\}) \cap E$ is infinite
So if U is open and $U \cap E' \neq \emptyset$ then $U \cap E$ is infinite.

Note: For any metric space (X, d)
$$\overline{B_r(p)} \subseteq \{q \in X : d(p,q) \leq r\}$$
Proof is an exercise

Thm 2.43: If $P \subseteq \mathbb{R}^k$ is perfect $(P' = P)$ and $P \neq \emptyset$ then P is uncountable.

Pf: Since $P' = P \neq \emptyset$, we must have that P is infinite.
Towards a contra, suppose
$$P = \{\vec{x}_0, \vec{x}_1, \vec{x}_2, \dots\}.$$
We will inductively build sets V_n, $n \in \mathbb{N}$, satisfying:
1. V_n is open
2. $V_n \cap P \neq \emptyset$
3. when $n \geq 1$, $\overline{V_n} \subseteq V_{n-1}$
4. when $n \geq 1$, $\vec{x}_{n-1} \notin V_n$

To begin, set $V_0 = B_1(\vec{x}_0)$. Now inductively assume that $V_0, \dots, V_n$ have been defined.
1 and 2 imply that $V_n \cap P$ is infinite. So we can pick $\vec{y} \in V_n \cap P$ with $\vec{y} \neq \vec{x}_n$.

$r>0$.

Let $r < d(\vec{y}, \vec{x}_n)$ be small enough so that $B_r(\vec{y}) \subseteq V_n$. Now set $V_{n+1} = B_{r/2}(\vec{y})$.

Set $K_n = \overline{V_n} \cap P$. Then K_n is compact and nonempty and $K_{n+1} \subseteq K_n$, so by finite intersection property (Thm. 2.36) there is $\vec{z} \in \bigcap_{n\in\mathbb{N}} K_n$. Each $K_n \subseteq P$ so $\vec{z} \in P$. Hence there is $n \in \mathbb{N}$ with $\vec{z} = \vec{x}_n \in P$. But (4) $\vec{z} = \vec{x}_n \notin K_{n+1}$, a contradiction. $\square$

Cor: For all $a<b \in \mathbb{R}$, $[a,b]$ is perfect hence uncountable. Similarly $\mathbb{R}$ is uncountable.

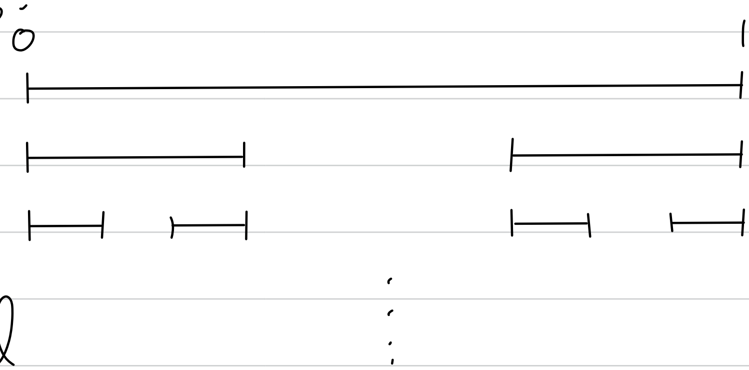
Ex: Build a seq. of sets:

$E_0 = [0,1]$

$E_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$

$E_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$

$\vdots$



Keep on removing the middle-third from each interval.

$$C = \bigcap_{n\in\mathbb{N}} E_n \text{ is the Cantor set}$$

- Each E_n is compact, so C is nonempty and compact
- E_n contains no interval of length greater than 3^{-n}, so C contains no intervals
- Each endpoint of each E_n is in C, and these endpoints are dense in C, so C is perfect.
- Since C perfect, it is uncountable.

Review

Two properties defining sup's (also inf's)
lub property
— know when sup's and inf's exist in $\mathbb{R}$ (Thm. 1.11)
Ordered fields
Properties of $\mathbb{R}$: archimedean, density of $\mathbb{Q}$ (Thm. 1.20)
existence and uniqueness of roots (Thm. 1.21)
$\mathbb{C}$, $\mathbb{R}^k$
Cardinality via bijections
Operations that preserve being countable
Diagnolization method for showing set is uncountable (Thm. 2.14)
Definition of metric spaces
Triangle inequality
Definition of limit points, closed sets
Definition of interior points, open sets
E open $\iff$ E^c closed.

Prove $\overline{B_r(p)} \subseteq \{q \in X : d(p,q) \leq r\}$

Pf: Clearly $B_r(p) \subseteq \{q \in X : d(p,q) \leq r\}$. So suffices to check $B_r(p)' \subseteq \{q \in X : d(p,q) \leq r\}$. Let $x \in B_r(p)'$. Towards a contra, suppose $d(p,x) > r$. Set $R = d(p,x) - r$. Then $R > 0$ and since $x \in B_r(p)'$ there is $y \in (B_R(x) \setminus \{x\}) \cap B_r(p)$. Then

$$\begin{aligned} d(p,x) &\leq d(p,y) + d(y,x) \\ &< r + R = d(p,x) \end{aligned}$$

So $d(p,x) < d(p,x)$, a contradiction. Thus $d(p,x) \leq r$ and $x \in \{q \in X : d(p,q) \leq r\}$. $\square$

If (X,d) is as in Ch. 2 Prob 10 (HW 3) and $r=1$ then

$$B_r(p) = \{p\} = \overline{B_r(p)}$$

but $\{q \in X : d(p,q) \leq 1\} = X$

Ch. 1 Prob 15

Cauchy-Schwarz

$$0 \leq \sum_{i=1}^{k} |Ba_i - Cb_i|^2 =$$
$$=$$
$$=$$
$$= B(AB - |C|^2)$$

Equality iff $\forall i$ $Ba_i - Cb_i = 0$

$$Ba_i = Cb_i$$
$$a_i = \frac{C}{B} \cdot b_i$$

Lecture 12 Oct 30

HW 4 due Monday 9:00 PM

Defn: A, B subsets of metric space (X,d) are separated if $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$.
$E \subseteq X$ is connected if E is not the union of two nonempty separated sets

Note: Separated is stronger than disjoint.
- $(0,1)$ and $(1,2)$ are separated (and disjoint)
- $(0,1]$ and $(1,2)$ are not separated but disjoint

Thm 2.47: $E \subseteq \mathbb{R}$ is connected iff
$\forall x \leq y \in E \quad [x,y] \subseteq E$.

Pf: Assume there are $x \leq y \in E$ with $[x,y] \not\subseteq E$.
Pick $x < z < y$ with $z \notin E$. Set
$A = E \cap (-\infty, z)$, $\quad B = E \cap (z, +\infty)$
Then A, B are nonempty $(x \in A,\ y \in B)$
And since $\bar{A} \subseteq (-\infty, z]$ and $\bar{B} \subseteq [z, +\infty)$,
A and B are separated. Also $A \cup B = E$
so E is not connected.

Next assume E is not connected. Say A, B nonempty, separated and $E = A \cup B$.
Pick $x \in A$ and $y \in B$. By swapping A and B we can assume $x < y$. Set

$$\underline{z} = \sup A \cap [x,y]$$

Then $z \in \overline{A \cap [x,y]} \subseteq \bar{A}$ hence $z \notin B$.
If $z \notin A$ then $z \notin E$ hence $[x,y] \nsubseteq E$ as $z \in [x,y] \setminus E$.
If $z \in A$ then $z \notin \bar{B}$. Since $z \in \overline{(z,y]}$ we must have $(z,y] \nsubseteq B$. So there is $z' \in (z,y] \setminus B$. Also $z' \notin A$ since $z' > z$.
Thus $z' \in [x,y] \setminus (A \cup B) = [x,y] \setminus E$. So $[x,y] \nsubseteq E$.
$\square$

Defn: A seq. $(p_n)_{n \in \mathbb{N}}$ in a metric space (X,d) converges if there is $p \in X$ with

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \ d(p_n, p) < \varepsilon$$

In this case we say $(p_n)_{n \in \mathbb{N}}$ converges to p or has limit p and write $p_n \to p$ or $\lim_{n \to \infty} p_n = p$.
If $(p_n)_{n \in \mathbb{N}}$ does not converge we say it diverges

Defn: • The range of $(p_n)_{n \in \mathbb{N}}$ is $\{p_n : n \in \mathbb{N}\}$
• $(p_n)_{n \in \mathbb{N}}$ is bounded if the range is bounded.

Thm 3.2:

(A) (p_n) converges to p iff $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N\ p_n \in B_\varepsilon(p)$

(B) If (p_n) converges to p and p' then $p = p'$

(C) (p_n) converges $\Rightarrow$ (p_n) bounded

(D) If $E \subseteq X$ and $p \in E'$ then $\exists$ seq (p_n) in E and $p_n \to p$

(E) If $\forall n\ p_n \in E$ and $p_n \to p$ then $p \in \overline{E}$

Pf: (A) This follows from fact $d(p_n, p) < \varepsilon \Leftrightarrow p_n \in B_\varepsilon(p)$

(B) Let $\varepsilon > 0$. Pick N, N' with
$\forall n \geq N\ d(p_n, p) < \varepsilon/2$ and $\forall n \geq N'\ d(p_n, p') < \varepsilon/2$
Then using $n = \max(N, N')$ we obtain
$$d(p, p') \leq d(p, p_n) + d(p_n, p') < \varepsilon/2 + \varepsilon/2 = \varepsilon$$
Since ε was arbitrary, $d(p, p') = 0$ and $p = p'$.

(C) Say $p_n \to p$. Pick N with $\forall n \geq N\ d(p_n, p) < 1$.
Set $r = \max(1, d(p_0, p), d(p_1, p), \dots, d(p_{N-1}, p))$
Then $\forall n \in \mathbb{N}\ d(p_n, p) \leq r$ so $\{p_n : n \in \mathbb{N}\}$ is bounded.

(D) For each $n \in \mathbb{Z}_+$ pick $p_n \in E \cap B_{1/n}(p)$. Given $\varepsilon > 0$ pick $N \in \mathbb{Z}_+$ with $\frac{1}{N} < \varepsilon$ $(N\varepsilon > 1)$. For $n \geq N$
$$d(p_n, p) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$
Thus $p_n \to p$.

(E) If $p \in E$ then we are done $(p \in \overline{E})$.
Assume $p \notin E$. Then for every $r > 0$ there is n with $p_n \in B_r(p) \cap E = (B_r(p) \setminus \{p\}) \cap E$.
Thus $(B_r(p) \setminus \{p\}) \cap E \neq \emptyset$ so $p \in E'$. $\square$

Thm 3.3: Suppose $(s_n)_{n\in\mathbb{N}}$, $(t_n)_{n\in\mathbb{N}}$ are seq's in $\mathbb{C}$ with $s_n \to s$ and $t_n \to t$. Then

1. $\lim_{n\to\infty}(s_n+t_n) = s+t$
2. $\forall c\in\mathbb{C}$ $\lim_{n\to\infty}(s_n+c) = s+c$, $\lim_{n\to\infty}(c\cdot s_n) = c\cdot s$
3. $\lim_{n\to\infty} s_n t_n = st$
4. $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ if $s\neq 0$ and $\forall n\in\mathbb{N}$ $s_n\neq 0$.
5. $(\forall n\in\mathbb{N}$ $s_n, t_n\in\mathbb{R}$, $s_n\leq t_n) \Rightarrow s\leq t$

Pf: ① For $\varepsilon>0$ pick N_1, N_2 with

$\forall n\geq N_1$ $|s_n-s|<\varepsilon/2$, $\forall n\geq N_2$ $|t_n-t|<\varepsilon/2$

Then for $n\geq \max(N_1,N_2)$

$|(s_n+t_n)-(s+t)| \leq |s_n-s|+|t_n-t| < \varepsilon$

Thus $s_n+t_n \to s+t$.

② Exercise

③ $s_n t_n = (s+(s_n-s))(t+(t_n-t))$

$= st + t(s_n-s)+s(t_n-t)+(s_n-s)(t_n-t)$

To see $(s_n-s)(t_n-t)\to 0$ let $\varepsilon>0$ and pick N_1, N_2 with

$\forall n\geq N_1$ $|s_n-s|<\sqrt{\varepsilon}$, $\forall n\geq N_2$ $|t_n-t|<\sqrt{\varepsilon}$.

Then for $n\geq\max(N_1,N_2)$

$|(s_n-s)(t_n-t)-0| = |s_n-s|\cdot|t_n-t|<\varepsilon$

Thus $(s_n-s)(t_n-t)\to 0$. So by ① and ②

$$\begin{aligned}\lim(s_n t_n) &= \lim\Big((s+(s_n-s))(t+(t_n-t))\Big)\\ &= \lim\big(st+t(s_n-s)+s(t_n-t)+(s_n-s)(t_n-t)\big)\\ &= st+t\cdot\lim(s_n-s)+s\cdot\lim(t_n-t)+\lim(s_n-s)(t_n-t)\\ &= st+0+0+0 = st\end{aligned}$$

Proof of ④ and ⑤ next class...

Lecture 13 Nov 2

HW 4 due today
HW 5 due Friday

(4) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ if $s \neq 0$ and $\forall n \in \mathbb{N}$ $s_n \neq 0$.
(5) $(\forall n \in \mathbb{N}\ s_n, t_n \in \mathbb{R},\ s_n \leq t_n) \Rightarrow s \leq t$

Pf: (4) Choose m with $\forall n \geq m$ $|s_n - s| < \frac{1}{2}|s|$.
Then for $n \geq m$ $|s_n| > \frac{1}{2}|s|$. Now let $\varepsilon > 0$ and pick $N \geq m$ with $\forall n \geq N$ $|s_n - s| < \frac{1}{2}|s|^2 \varepsilon$.
Then for $n \geq N$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|s_n s|} < \frac{1}{\frac{1}{2}|s|^2} \cdot |s - s_n| < \varepsilon$$

Thus $\frac{1}{s_n} \to \frac{1}{s}$.

(5) For every n, $t_n - s_n$ lies in the closed set $[0, +\infty) \subseteq \mathbb{R} \subseteq \mathbb{C}$. Thus $t - s = \lim_{n\to\infty}(t_n - s_n) \in [0, +\infty)$ so $s \leq t$. $\square$

Thm 3.4: (A) If $\vec{x}_n = (\alpha_{1_n}, \alpha_{2_n}, \dots, \alpha_{k_n}) \in \mathbb{R}^k$ then $\vec{x}_n \to \vec{x} = (\alpha_1, \dots, \alpha_k)$ iff $\forall 1 \leq i \leq k$ $\alpha_{i_n} \to \alpha_i$

(B) Let $(\vec{x}_n), (\vec{y}_n)$ be seq.'s in $\mathbb{R}^k$ with $\vec{x}_n \to \vec{x}$, $\vec{y}_n \to \vec{y}$. Let (β_n) be seq in $\mathbb{R}$ with $\beta_n \to \beta$. Then

- $\vec{x}_n + \vec{y}_n \to \vec{x} + \vec{y}$
- $\vec{x}_n \cdot \vec{y}_n \to \vec{x} \cdot \vec{y}$
- $\beta_n \vec{x}_n \to \beta \vec{x}$

Pf: (A) follows from the following inequalities:

- $\forall 1 \le i \le k \quad |\alpha_{i_n} - \alpha_i| \le |\vec{x}_n - \vec{x}|$
- $|\vec{x}_n - \vec{x}| = \left(\sum_{i=1}^{k} |\alpha_{n_i} - \alpha_i|^2\right)^{1/2} \le \sqrt{k} \cdot \left(\max_{1\le i \le k} |\alpha_{n_i} - \alpha_i|\right)$

(B) follows from (A) and previous theorem. $\square$

Defn: If $n_1 < n_2 < n_3 < \cdots$ are integers in $\mathbb{N}$ (or $\mathbb{Z}_+$) then $(p_{n_i})_{i\in\mathbb{N}}$ (or $(p_{n_i})_{i\in\mathbb{Z}_+}$) is called a subsequence of (p_n). If (p_{n_i}) converges, its limit is called a subsequential limit of (p_n).

Thm: A point q in a metric space (X,d) is a subseq. limit of $(p_n)_{n\in\mathbb{N}}$ iff
$\forall r > 0 \quad \{n \in \mathbb{N} : p_n \in B_r(q)\}$ is infinite

Pf: First assume (p_{n_i}) is subseq with $p_{n_i} \to q$. Let $r>0$. Pick N with $\forall i \ge N \quad d(p_{n_i}, q) < r$. Then $\{n_N, n_{N+1}, n_{N+2}, \ldots\} \subseteq \{n \in \mathbb{N} : p_n \in B_r(q)\}$ thus $\{n \in \mathbb{N} : p_n \in B_r(q)\}$ is infinite.

Now assume $\forall r > 0 \quad \{n \in \mathbb{N} : p_n \in B_r(q)\}$ is infinite. Pick any n_0 with $p_{n_1} \in B_1(q)$. Once $n_0 < \cdots < n_{i-1}$ have been defined, pick $n_i > n_{i-1}$ with $p_{n_i} \in B_{1/i}(q)$. This defines a subseq $(p_{n_i})_{i\in\mathbb{N}}$. Let $\varepsilon > 0$. Pick N with $\frac{1}{N} < \varepsilon$. Then for $i \ge N$ $d(p_{n_i}, q) < \frac{1}{i} \le \frac{1}{N} < \varepsilon$ since $p_{n_i} \in B_{1/i}(q)$. Thus $p_{n_i} \to q$. $\square$

Cor: If $q \in \{p_n : n \in \mathbb{N}\}'$ then q is a subseq. limit of (p_n).

Pf: Let $r > 0$. Set $I = \{n \in \mathbb{N} : p_n \in B_r(q)\}$.
Then $\{p_i : i \in I\} = \{p_n : n \in \mathbb{N}\} \cap B_r(q)$
and the right-hand set is infinite since $q \in \{p_n : n \in \mathbb{N}\}'$. So $\{p_i : i \in I\}$ is infinite, so I must be infinite as well. $\square$

Thm 3.6: (A) If (p_n) seq. in compact metric space (X,d) then (p_n) has a subseq. limit.
(B) Every bounded seq. in $\mathbb{R}^k$ has a convergent subseq.

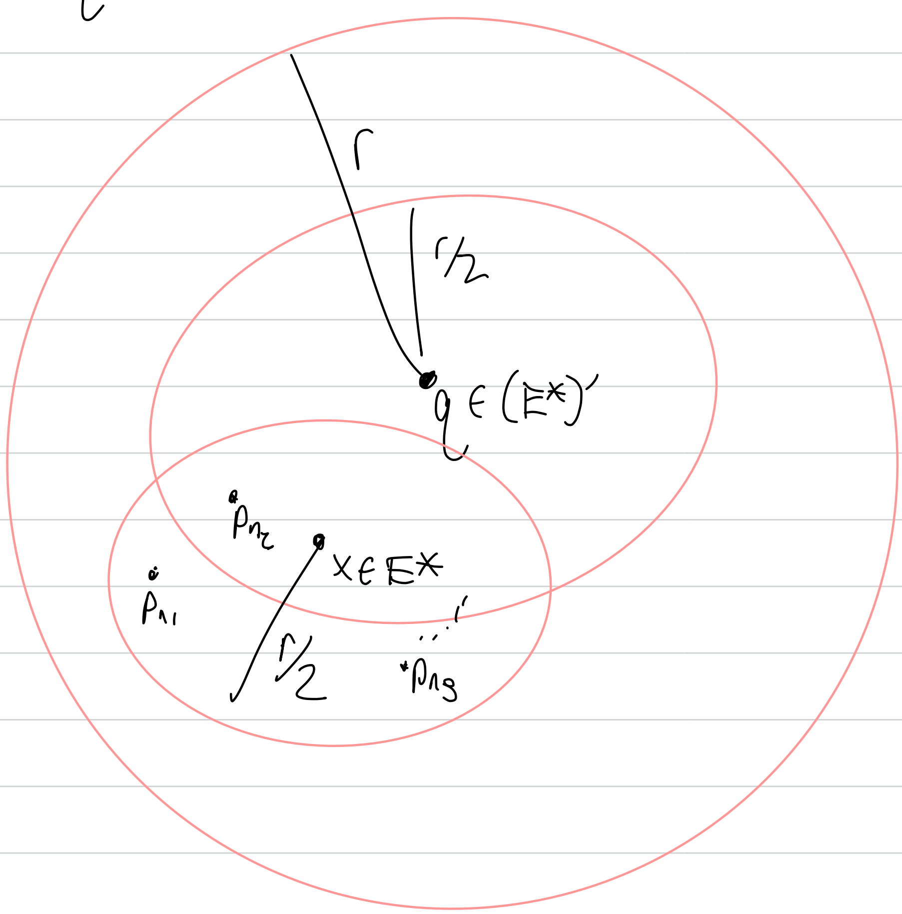
Pf: (A) Set $E = \{p_n : n \in \mathbb{N}\}$. If E is finite then there must be some $p \in E$ and $n_1 < n_2 < n_3 < \cdots$ with $\forall i$ $p_{n_i} = p$. So $p_{n_i} \to p$.
If E is infinite then $E' \neq \emptyset$ by earlier theorem. Thus (p_n) has a subseq limit by previous corollary.
(B) follows from (A) and Heine-Borel theorem. $\square$

Thm 3.7 The set of all subseq limits of (p_n) is a closed set.

Pf: Let E^* be set of all subseq. limits of (p_n). Let $q \in (E^*)'$. Let $r>0$. Since $q \in (E^*)'$ we can pick $x \in (B_{r/2}(q) \setminus \{q\}) \cap E^*$. Notice that $B_{r/2}(x) \subseteq B_r(q)$

(if $w \in B_{r/2}(x)$ then $d(q,w) \leq d(q,x) + d(x,w) < r$ and hence $w \in B_r(q)$),

Since $x \in E^*$, $\{n \in \mathbb{N} : p_n \in B_{r/2}(x)\}$ is infinite. Therefore $\{n \in \mathbb{N} : p_n \in B_r(q)\}$ is infinite since it contains $\{n \in \mathbb{N} : p_n \in B_{r/2}(x)\}$. By prior theorem $q \in E^*$. We conclude E^* is closed. $\square$



Defn: A seq (p_n) in a metric space (X,d) is **Cauchy** if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n,m \geq N \ d(p_n, p_m) < \varepsilon$$

Defn: The **diameter** of nonempty $E \subseteq X$ is

$$\operatorname{diam} E = \sup \{ d(p,q) : p,q \in E \}$$

if the supremum exists (otherwise $\operatorname{diam} E = \infty$)

Obs: (p_n) Cauchy $\Leftrightarrow \lim_{n\to\infty} \operatorname{diam} \{p_n, p_{n+1}, p_{n+2}, \ldots\} = 0$

Lecture 14 Nov 4

HW 5 due Friday

Thm 3.10: Let (X,d) be metric space,

(A) If $\emptyset \neq E \subseteq X$ and $\operatorname{diam} E$ exists then $\operatorname{diam} \overline{E} = \operatorname{diam} E$

(B) If K_n compact, nonempty, $K_n \supseteq K_{n+1}$, and $\lim_{n\to\infty} \operatorname{diam} K_n = 0$ then $\bigcap_{n \in \mathbb{N}} K_n$ is a singleton

Pf: (A) Clearly $\operatorname{diam} \overline{E} \geq \operatorname{diam} E$ since $\overline{E} \supseteq E$.
Let $p, q \in \overline{E}$. Let $\varepsilon > 0$ and pick $p', q' \in E$ with $d(p,p'), d(q,q') < \varepsilon$. Then

$$d(p,q) \leq d(p,p') + d(p',q') + d(q',q)$$
$$< 2\varepsilon + \operatorname{diam} E.$$

This holds for all $\varepsilon > 0$ so $d(p,q) \leq \operatorname{diam} E$.
This holds for all $p, q \in \overline{E}$ so $\operatorname{diam} \overline{E} \leq \operatorname{diam} E$.

(B) Set $K = \bigcap_{n \in \mathbb{N}} K_n$. Then $K \neq \emptyset$ by Thm 2.36 and $\operatorname{diam} K \leq \operatorname{diam} K_n$ for all n since $K \subseteq K_n$.
Thus $\operatorname{diam} K = 0$ and hence K consists of a single point $\square$

Thm 3.11: ① For any metric space (X,d) if seq (p_n) converges, then (p_n) is Cauchy.
② If (X,d) is compact and (p_n) Cauchy then (p_n) converges.
③ In $\mathbb{R}^k$ every Cauchy sequence converges (Cauchy Criterion)

Pf: ① Say $p_n \to p$. Let $\varepsilon > 0$. Pick N with $\forall n \geq N$ $d(p_n, p) < \varepsilon/2$. Then for $n, m \geq N$ we have

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Thus (p_n) is Cauchy.

② Set $E_n = \{p_n, p_{n+1}, p_{n+2}, \dots\}$. Since (p_n) is Cauchy, $\lim_{n\to\infty} \operatorname{diam} \overline{E_n} = 0$ by Thm 3.10 Ⓐ. X is cmpct so each $\overline{E_n}$ cmpct and $\overline{E_n} \supseteq \overline{E_{n+1}}$. So by Thm 3.10 Ⓑ $\bigcap_{n\in\mathbb{N}} \overline{E_n} = \{p\}$ for some $p \in X$. Now let $\varepsilon > 0$ and pick N with $\operatorname{diam} \overline{E_N} < \varepsilon$. Then for $n \geq N$, we have $p_n \in E_n \subseteq \overline{E_N}$ and $p \in \overline{E_N}$ so $d(p_n, p) \leq \operatorname{diam} \overline{E_N} < \varepsilon$. Thus $p_n \to p$.

③ Say $(\vec{x}_n)$ Cauchy seq in $\mathbb{R}^k$. Pick N with $\operatorname{diam}\{\vec{x}_N, \vec{x}_{N+1}, \dots\} < 1$. Then for $n \geq N$

$$|\vec{x}_n| \leq |\vec{x}_N| + |\vec{x}_n - \vec{x}_N| < |\vec{x}_N| + 1$$

So $\{\vec{x}_0, \vec{x}_1, \dots\} \subseteq B_r(\vec{0})$ where $r = 1 + \min(|\vec{x}_0|, \dots, |\vec{x}_N|)$.
By Heine-Borel theorem, $(\vec{x}_n)$ is a seq in the compact set $\overline{B_r(\vec{0})}$, hence converges by ②. ☐

Defn: A metric space (X,d) is <u>complete</u> if every Cauchy sequence converges.

Ex: Compact spaces, Euclidean spaces, closed subsets of these are complete.

Non-ex: $\mathbb{Q}$ is not complete

Fact: $\mathbb{R}$ is the smallest complete metric space containing $\mathbb{Q}$ (Cauchy construction)

Defn: A seq. (s_n) in $\mathbb{R}$ is

- <u>monotone increasing</u> if $\forall n$ $s_n \leq s_{n+1}$
- <u>monotone decreasing</u> if $\forall n$ $s_n \geq s_{n+1}$
- <u>monotone</u> if either of the above.

Thm 3.14: Suppose (s_n) is monotone. Then (s_n) converges iff (s_n) is bounded

Pf: $(\Rightarrow)$ follows by Thm. 3.2

$(\Leftarrow)$ Let's say (s_n) is monotone increasing (the other case is similar). Since (s_n) is bounded, $s = \sup\{s_n : n \in \mathbb{N}\}$ exists. Let $\varepsilon > 0$. Since $s-\varepsilon$ is not an upper bound to $\{s_n : n \in \mathbb{N}\}$, so there is N with $s_N > s-\varepsilon$. Then for $n \geq N$ $s-\varepsilon < s_N \leq s_n \leq s$ hence $|s-s_n| < \varepsilon$.

We conclude $s_n \to s$. $\square$

Defn: For a seq. (s_n) in $\mathbb{R}$ we write

- $\lim_{n\to\infty} s_n = +\infty$ or $s_n \to +\infty$ if $\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N\ s_n \geq M$
- $\lim_{n\to\infty} s_n = -\infty$ or $s_n \to -\infty$ if $\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N\ s_n \leq M$

Note: When either of the above holds, (s_n) still diverges ($\mathbb{R} \cup \{-\infty, +\infty\}$ is not a metric space)

Defn: Let (s_n) be seq. in $\mathbb{R}$. Let E be set of all subsequential limits of (s_n) (including $+\infty, -\infty$ if appropriate).

- The upper limit or limit supremum of (s_n), denoted $\limsup_{n\to\infty} s_n$, is $\sup E \in \mathbb{R} \cup \{-\infty, +\infty\}$
- The lower limit or limit infimum of (s_n), denoted $\liminf_{n\to\infty} s_n$, is $\inf E \in \mathbb{R} \cup \{+\infty, -\infty\}$

Obs: $E \neq \emptyset$ by Bolzano-Weierstrauss. (Exercise)

Thm 3.17: Let (s_n), E be as above. Then

(A) $\limsup_{n\to\infty} s_n \in E$

(B) If $x > \limsup_{n\to\infty} s_n$ then $\exists N \in \mathbb{N}\ \forall n \geq N\ s_n < x$

Moreover, $\limsup_{n\to\infty} s_n$ is the unique extended real number with these properties.

Note: Similar result holds for $\liminf_{n\to\infty} s_n$.

Proof next time...

Lecture 15 Nov 6

HW 5 due today

Pf of Thm 3.17:

Ⓐ If $\limsup s_n \in \mathbb{R}$ then
$\limsup s_n = \sup E \in \overline{E} = E$ by Thm's 3.7 and 2.28.
If $\limsup s_n = +\infty$ then E is not bounded above by anything in $\mathbb{R}$, hence $\{s_n : n \in \mathbb{N}\}$ is not bounded above (in $\mathbb{R}$) so there is subseq. (s_{n_k}) with $s_{n_k} \to +\infty$. Thus $\limsup s_n = +\infty \in E$.
If $\limsup s_n = -\infty$ then $E = \{-\infty\}$ hence $\limsup s_n \in E$.

Ⓑ Towards contra, suppose $s_n \geq x$ for infinitely-many n. Then (s_n) has a subseq in $[x, +\infty)$, hence has a subsequential limit $y \in [x, +\infty]$. Thus $\limsup s_n = \sup E \geq y \geq x$ (since $y \in E$), contradicting $x > \limsup s_n$.

Lastly, suppose $p < q$ both satisfy Ⓐ and Ⓑ. Choose $p < x < q$. Applying Ⓑ to p and x, we have $\exists N\ \forall n \geq N\ s_n < x$. It follows every subseq. limit of (s_n) is in $[-\infty, x]$. So $E \subseteq [-\infty, x]$. Thus q cannot satisfy Ⓐ, contradiction. $\square$

Ex: For $s_n = (-1)^n\left(1 + \frac{1}{2^n}\right)$, $\limsup s_n = 1$, $\liminf s_n = -1$

Obs: $\lim s_n$ exists and equals s
iff $\limsup s_n = s = \liminf s_n$

Thm 3.19: If $\forall n$ $s_n \geq t_n$ then
$$\limsup s_n \geq \limsup t_n \quad \text{and} \quad \liminf s_n \geq \liminf t_n$$

Pf: Exercise

Defn: For $n, k \in \mathbb{N}$, $0 \leq k \leq n$
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$
(this is pronounced "n choose k")

Binomial Theorem: For $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Pf Sketch: It is easy to check that
$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

Given this, easy to prove binomial theorem by induction. $\square$

Thm 3.20: (A) If $p>0$ $\lim_{n\to\infty} \frac{1}{n^p} = 0$

(B) If $p>0$, $\lim_{n\to\infty} \sqrt[n]{p} = 1$

(C) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

(D) If $p>0$ and $\alpha \in \mathbb{R}$, $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

(E) If $z \in \mathbb{C}$ and $|z|<1$ then $\lim_{n\to\infty} z^n = 0$

Pf: (A) $\left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} < \varepsilon$ whenever $\left(\frac{1}{\varepsilon}\right)^{1/p} < n$.

Follows from archimedean principle $\frac{1}{n^p} \to 0$

(B) Clear if $p=1$.

Assume $p>1$. Set $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$ and by Binomial Thm

$$1 + nx_n \leq (1+x_n)^n = p$$

So $0 < x_n \leq \frac{p-1}{n}$ and thus $x_n \to 0$

If $0<p<1$ then by Thm 3.3

$$1 = \frac{1}{1} = \frac{1}{\lim \sqrt[n]{\frac{1}{p}}} = \lim \frac{1}{\sqrt[n]{\frac{1}{p}}} = \lim \sqrt[n]{p}$$

(C) Set $x_n = \sqrt[n]{n} - 1$. Then $x_n > 0$ and by Binomial Thm

$$\binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2 \leq (1+x_n)^n = n$$

So $0 < x_n \leq \sqrt{\frac{2}{n-1}}$ thus $x_n \to 0$ (when $\frac{2}{n-1} < \varepsilon^2$ we have $\sqrt{\frac{2}{n-1}} < \varepsilon$)

(D) Fix $k \in \mathbb{N}$ with $k > \alpha$. When $n > 2k$ by Binom. Thm

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k > \left(\frac{n}{2}\right)^k \cdot \frac{p^k}{k!} = \frac{n^k p^k}{2^k \cdot k!}$$

Thus $0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k \cdot k!}{p^k} \cdot \frac{1}{n^{k-\alpha}}$

Since $k - \alpha > 0$, $\frac{2^k \cdot k!}{p^k} \cdot \frac{1}{n^{k-\alpha}} \to 0$ by (A) and Thm 3.3

(E) Apply (D) with $\alpha = 0$ and $p = \frac{1}{|z|} - 1$

we find $|z|^n \to 0$. Since $|z^n| = |z|^n$,

we obtain $z^n \to 0$ $\square$

Defn: Given a seq (a_n) in $\mathbb{C}$ we write $\sum_{n=p}^{q} a_n$ for $a_p + a_{p+1} + \cdots + a_q$ (for $p \leq q \in \mathbb{Z}$).

We associate to (a_n) the partial sums $s_n = \sum_{k=0}^{n} a_k$.

The expressions $a_0 + a_1 + a_2 + \cdots$ and $\sum_{n \in \mathbb{N}} a_n$

are called (infinite) series and denote the value $\lim_{n \to \infty} s_n$ when it exists. We say $\sum_{n \in \mathbb{N}} a_n$ converges/diverges if (s_n) converges/diverges.

Series and seq's are closely connected.

Thm 3.22: $\sum a_n$ converges iff $\forall \varepsilon > 0 \; \exists N \; \forall n, m \geq N \; \left| \sum_{k=n}^{m} a_k \right| < \varepsilon$

Pf: This follows from Cauchy criterion (Thm 3.11) and $|s_m - s_n| = \left| \sum_{k=n}^{m} a_k \right|$.

Thm 3.23: If $\sum a_n$ converges then $a_n \to 0$

Pf: Follows from Thm. 3.22 by using $m = n$.

Lecture 16 Nov 9

HW 6 due Friday

Wed is a university holiday

My office hours this week: Th 3-5, F 1-2

$\times$ Please read my email about Eastern Hemisphere Second Midterm, respond promptly if needed $\times$

Obs: Converse of Thm 3.23 is false:

$\frac{1}{n} \to 0$ but $\sum \frac{1}{n}$ diverges

Thm 3.24: If $a_n \geq 0$ then $\sum a_n$ converges iff its partial sums are bounded

Pf: $\forall n$ $a_n \geq 0$ implies partial sums $s_n = \sum_{k=0}^{n} a_k$ are mono. increasing. Apply Thm. 3.14 $\square$

Thm 3.25 (Comparison Test):

① If $|a_n| \leq c_n$ for $n \geq N$ and $\sum c_n$ converges then $\sum a_n$ converges

② If $a_n \geq d_n \geq 0$ and $\sum d_n$ diverges then $\sum a_n$ diverges

Pf: ① Given $\varepsilon > 0$ pick M with $\forall m \geq n \geq M$ $\sum_{k=n}^{m} c_k < \varepsilon$.

Then for $m \geq n \geq \max(N, M)$

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k < \varepsilon$$

So $\sum a_n$ converges by Thm. 3.22.

② Follows from ① (also follows from previous theorem) $\square$

Thm 3.26: If $z \in \mathbb{C}$ and $|z|<1$ then $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.
If $|z| \geq 1$ then $\sum z^n$ diverges.

Pf: Notice $(1-z)\cdot \sum_{k=0}^{n} z^k = 1-z^{n+1}$ so
$s_n = \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$ thus $\lim_{n\to\infty} s_n = \frac{1}{1-z}$
when $|z|<1$. When $|z| \geq 1$ we have
$z^n \not\to 0$ hence $\sum z^n$ diverges (by Thm. 3.23). $\square$

Defn: $\sum_{n=0}^{\infty} z^n$ is called a geometric series

Thm 3.27: Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$.
Then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Pf: For both series, they converge iff their partial sums are bounded (Thm 3.24).
Set $s_n = a_1 + a_2 + a_3 + \cdots + a_n$
$t_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$
When $n < 2^k$

$$\begin{aligned} s_n &\leq a_1 + (a_2+a_3) + (a_4+a_5+a_6+a_7) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = t_k \end{aligned}$$

And when $n > 2^k$

$$\begin{aligned} 2s_n &\geq a_1 + 2a_2 + 2(a_3+a_4) + \cdots + 2(a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\geq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = t_k \end{aligned}$$

Thus (s_n) is bounded iff (t_k) is bounded. $\square$

Thm 3.28: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$, diverges if $p \le 1$.

Pf: When $p \le 0$ series diverges since $\frac{1}{n^p} \not\to 0$.
Assume $p>0$. By previous thm. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff
$$\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} \left(2^{1-p}\right)^k \text{ converges.}$$
This is a geometric series, so converges iff $2^{1-p} < 1$ iff $p>1$. $\square$

Note: We haven't learned about log yet, but for the sake of example lets pretend we know what it is.

Thm 3.29: $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p>1$, diverges if $p \le 1$.

Pf: When $p \le 0$ series diverges because $\frac{1}{n} \le \frac{1}{n(\log n)^p}$ for $n \ge 11$.
(use comparison test).
Assume $p>0$. Terms are monotone decreasing and positive, so by Thm 3.27 convergence happens iff
$$\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k(\log 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{k^p(\log 2)^p} = \frac{1}{(\log 2)^p} \cdot \sum_{k=1}^{\infty} \frac{1}{k^p}$$
converges. By Thm 3.28 this happens iff $p>1$. $\square$

Defn: $e = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.71828$ $\quad (0! = 1,\ n! = n(n-1)\cdots 3\cdot 2\cdot 1)$

Obs: $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$ so $\sum \frac{1}{n!}$ converges by comparison with geometric series $\sum_{n=0}^{\infty} \frac{1}{2^{n-1}} = 2\cdot\sum_{n=0}^{\infty} \frac{1}{2^n}$

Thm 3.31: $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

- Proof next class...

Lecture 17 Nov 13

HW 6 due today

Email me by tomorrow if you can't take 2^{nd} midterm at these times:

- class time 11:00–11:50 AM Wed Nov 25 (San Diego time)
- 12 hours later 11:00–11:50 PM Wed Nov 25 (San Diego time)

Thm 3.31: $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$ (by definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}$)

Pf: Set $s_n = \sum_{k=0}^{n} \frac{1}{k!}$ and $t_n = \left(1+\frac{1}{n}\right)^n$.

By Binomial Theorem

$$t_n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2}\cdot\frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}\cdot\frac{1}{n^k} + \cdots$$

$$\cdots + \frac{n(n-1)\cdots(n-(n-1))}{n!}\cdot\frac{1}{n^n}$$

$$* \quad = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) + \cdots$$

$$\cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$\le s_n \le e$$

So $\limsup t_n \le e$.

Now fix $m \in \mathbb{N}$. If $n \ge m$ then (by $*$)

$$t_n \ge 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)$$

Holding m fixed, take liminf over n on both sides to get

$$\liminf t_n \ge 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!} = s_m$$

Taking lim as $m\to\infty$ we obtain

$$\liminf t_n \ge \lim s_m = e$$

Since $e \le \liminf t_n \le \limsup t_n \le e$ so all are equal and (t_n) converges to e. $\square$

Thm 3.32: e is irrational

Pf: Towards a contra, say $e = \frac{p}{q}$, $p, q \in \mathbb{N}$. Then

$$0 < e - s_q = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \frac{1}{(q+3)!} + \cdots$$
$$< \frac{1}{(q+1)!}\left(1 + \frac{1}{q+1} + \frac{1}{(q+1)^2} + \cdots\right)$$
$$= \frac{1}{(q+1)!} \cdot \left(\frac{1}{1 - \frac{1}{q+1}}\right) = \frac{1}{(q+1)!} \cdot \frac{q+1}{q} = \frac{1}{q!q}$$

So $0 < q!(e - s_q) < \frac{1}{q}$

By assumption, $q!e$ is an integer.

Also $q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right)$ is an integer.

So $q!(e - s_q)$ is an integer strictly between 0 and $\frac{1}{q}$, contradiction. $\square$

Fact: e is not algebraic

Thm 3.33 (Root Test): Consider a series $\sum a_n$ and set $\alpha = \limsup \sqrt[n]{|a_n|}$.

(A) If $\alpha < 1$, $\sum a_n$ converges

(B) If $\alpha > 1$, $\sum a_n$ diverges

(C) If $\alpha = 1$, no information

Pf: (A) Pick β with $\alpha < \beta < 1$ and choose N (Thm 3.17) with $\forall n \geq N$ $\sqrt[n]{|a_n|} < \beta$.

(converges to $\frac{1}{1-\beta}$) $\to$ $\sum \beta^n$ converges and $|a_n| < \beta^n$ for $n \geq N$

so $\sum a_n$ converges by comparison.

(B) If $\alpha > 1$ then there is a subseq (a_{n_k}) with $\forall k$ $|a_{n_k}| > 1$, So $a_n \not\to 0$ and thus $\sum a_n$ diverges

(C) $\alpha = 1$ for both $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$, the first diverges and the second converges. $\square$

Thm 3.34 (Ratio Test): Suppose $\forall n$ $a_n \neq 0$.
Then $\sum a_n$

1. converges if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$
2. diverges if $\exists N$ $\forall n \geq N$ $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$

Pf: (1) Pick $\limsup \left|\frac{a_{n+1}}{a_n}\right| < \beta < 1$ and pick N with $\forall n \geq N$ $\left|\frac{a_{n+1}}{a_n}\right| < \beta$. Then

$$|a_{N+1}| < \beta |a_N| \text{ and by induction}$$

$$|a_{N+k}| < \beta^k |a_N|,$$

meaning $|a_n| < \beta^{n-N} |a_N| = \beta^{-N} |a_N| \cdot \beta^n$ for $n \geq N$.

$\sum \beta^n$ converges so $\sum a_n$ converges by comparison.

(2) This is immediate since $a_n \not\to 0$.
$(0 < |a_n| \leq |a_{n+1}| \leq |a_{n+2}| \leq \cdots)$ $\square$

Note: Root test is ~~always~~ most of the time more accurate than the ratio test. But sometimes the root test is harder to evaluate.

Ex: For the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots$

$$\limsup \sqrt[n]{|a_n|} = \frac{1}{\sqrt{2}}, \quad \limsup \frac{a_{n+1}}{a_n} = \infty$$

Root test gives convergence, ratio test gives no info.

Thm 3.37: For any seq. (a_n), $\forall n\ a_n > 0$

$$\liminf \frac{a_{n+1}}{a_n} \underset{①}{\leq} \liminf \sqrt[n]{|a_n|} \underset{②}{\leq} \limsup \sqrt[n]{|a_n|} \underset{③}{\leq} \limsup \frac{a_{n+1}}{a_n}$$

Pf: ② is immediate. We will prove ③. ① is similar.

Pick $\beta > \limsup \frac{a_{n+1}}{a_n}$ and pick N with

$$\forall n \geq N \quad \frac{a_{n+1}}{a_n} < \beta$$

Then (as before) by induction for $n \geq N$

$a_n < \beta^{n-N} a_N$ so

$$\sqrt[n]{a_n} < \sqrt[n]{\beta^{n-N} a_N} = \sqrt[n]{\beta^{-N} a_N} \cdot \beta$$

So $\limsup \sqrt[n]{a_n} \leq \beta$.

$\beta > \limsup \frac{a_{n+1}}{a_n}$ was arbitrary, so

$$\limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n}. \qquad \square$$

Lecture 18 Nov 16

HW 7 due Friday

Second midterm next week at two times:

- classtime 11:00–11:50 AM Wed Nov 25
- 12 hours later 11:00–11:50 PM Wed Nov 25

Defn: For seq (c_n) in $\mathbb{C}$ and $z \in \mathbb{C}$, the series $\sum_{n=0}^{\infty} c_n z^n$ is called a power series

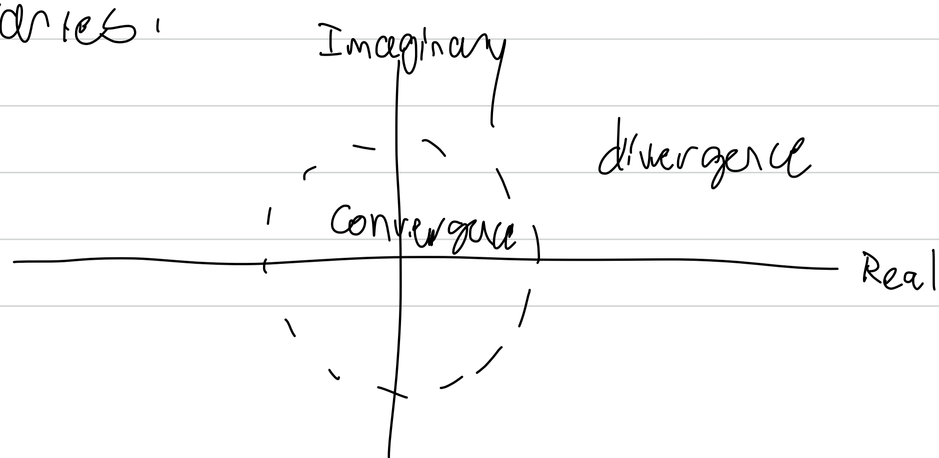
Note: Convergence depends on value of z

Thm 3.39: For a power series $\sum_{n=0}^{\infty} c_n z^n$ set $\alpha = \limsup \sqrt[n]{|c_n|}$ and $R = \frac{1}{\alpha}$ (if $\alpha = 0$ set $R = +\infty$, if $\alpha = +\infty$ set $R = 0$). Then $\sum c_n z^n$ converges when $|z| < R$ and diverges when $|z| > R$.

Pf: Apply root test:

$$\limsup \sqrt[n]{|c_n z^n|} = |z| \cdot \limsup \sqrt[n]{|c_n|} = \frac{|z|}{R} \qquad \square$$

Note: R is called the radius of convergence.
Convergence/divergence when $|z| = R$ is complicated and varies:



Ex: • For $\sum n^n z^n$, $R=0$
• For $\sum \frac{z^n}{n!}$, $R=\infty$
• For $\sum z^n$, $R=1$ and diverges when $|z|=1$ since $z^n \not\to 0$,
• For $\sum \frac{z^n}{n^2}$, $R=1$ and converges when $|z|=1$ (compare with $\sum \frac{1}{n^2}$)
• For $\sum \frac{z^n}{n}$, $R=1$, diverges when $z=1$ but converges if $|z|=1$ and $z\neq 1$ (Thm 3.44)

Recall from calculus: By integration by parts

$$\int_a^b fg\,dx = -\int_a^b Fg'\,dx + \Big[Fg\Big]_a^b$$

where $F'=f$.

Thm 3.41: For seq's (a_n), (b_n) set $A_{-1}=0$ and $A_n = \sum_{k=0}^{n} a_k$ for $n\geq 0$. Then for $0\leq p\leq q$

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Pf: $$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$

$$= \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \quad \square$$

Thm 3.42: If the partial sums of $\sum a_n$ are bounded and $b_0 \geq b_1 \geq b_2 \geq \cdots \geq 0$ with $\lim b_n = 0$ then $\sum a_n b_n$ converges

Pf: Set $A_{-1} = 0$, $A_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$.
Pick M with $\forall n$ $|A_n| \leq M$.
Let $\varepsilon > 0$ and pick N with $b_N < \frac{\varepsilon}{2M}$.
For $q \geq p \geq N$

$$\left|\sum_{n=p}^{q} a_n b_n\right| = \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right|$$
$$\leq \sum_{n=p}^{q-1} |A_n| \cdot (b_n - b_{n+1}) + |A_q| b_q + |A_{p-1}| b_p$$
$$\leq M\left(\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right)$$
$$= M(b_p - b_q + b_q + b_p)$$
$$= 2M b_p \leq 2M b_N < \varepsilon.$$

Thus $\sum a_n b_n$ converges by Cauchy criterion. $\square$

Thm 3.43 (Alternating Series Test):
Suppose $|c_1| \geq |c_2| \geq \cdots$, $c_{2m-1} \geq 0$ and $c_{2m} \leq 0$ for $m \geq 1$ and $\lim c_n = 0$. Then $\sum c_n$ converges

Pf: Apply above theorem with $a_n = (-1)^{n+1}$, $b_n = |c_n|$ $\square$

Thm 3.44: Suppose $\sum c_n z^n$ has radius of convergence 1, $c_0 \geq c_1 \geq \cdots$ and $\lim c_n = 0$. Then $\sum c_n z^n$ converges for all z with $|z|=1$ except possibly $z=1$.

Pf: Apply Thm 3.42 with $a_n = z^n$, $b_n = c_n$ and note if $|z|=1$ and $z \neq 1$ then

$$\left|\sum_{k=0}^{n} a_k\right| = \left|\sum_{k=0}^{n} z^k\right| = \left|\frac{1-z^{n+1}}{1-z}\right| \leq \frac{2}{|1-z|} \qquad \square$$

Defn: $\sum a_n$ converges absolutely if $\sum |a_n|$ converges. If $\sum a_n$ converges but $\sum |a_n|$ diverges, we say $\sum a_n$ converges non-absolutely

Thm 3.45: If $\sum a_n$ converges absolutely then it converges.

Pf Sketch: Apply Cauchy criterion and notice

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \qquad \square$$

Note: If $\forall n$ $a_n \geq 0$ then absolute convergence is same as convergence

Note: Comparison, root, and ratio test demonstrate absolute convergence

Lecture 19 Nov 18

HW 7 due Friday

2^{nd} Midterm next week at two times:

- Class time 11:00–11:50 AM Wed Nov 25
- 12 hours later 11:00–11:50 PM Wed Nov 25

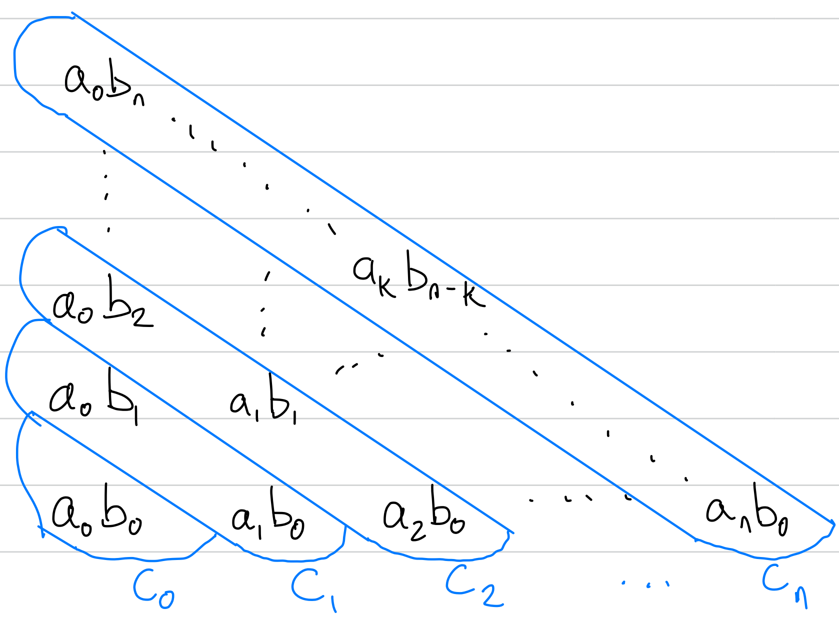
Thm 3.47 If $\sum a_n = A$ and $\sum b_n = B$ then $\sum (a_n + b_n) = A + B$ and $\forall c \in \mathbb{C}$ $\sum c \cdot a_n = c \cdot A$

Pf: Set $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$. Then $A_n + B_n = \sum_{k=0}^{n} (a_k + b_k)$.

So $A + B = \lim A_n + \lim B_n = \lim (A_n + B_n) = \sum (a_n + b_n)$.

and $c \cdot A = c \cdot \lim A_n = \lim c \cdot A_n = \sum c \cdot a_n$ $\square$

Defn: The (Cauchy) product of $\sum a_n$, $\sum b_n$ is $\sum c_n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$



Note: Motivation from "suspected" equalities

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \left(a_0 + a_1 z + a_2 z^2 + \cdots\right)\left(b_0 + b_1 z + b_2 z^2 + \cdots\right)$$

$$= (a_0 b_0) + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots$$

$$= c_0 + c_1 z + c_2 z^2 + \cdots = \sum_{n=0}^{\infty} c_n z^n$$

Note: It is not clear, and sometimes false, that

$$\sum c_n = \left(\sum a_n\right)\left(\sum b_n\right)$$

Ex: Suppose $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$.

$\sum a_n$, $\sum b_n$ converge (but not absolutely)

$$|c_n| = \left|\sum_{k=0}^{n} a_k b_{n-k}\right| = \left|\sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}}\right| \geq \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2}$$

$$(k+1)(n-k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \leq \left(\frac{n}{2}+1\right)^2$$

so $c_n \not\to 0$ and $\sum c_n$ diverges

Thm 3.50 (Mertens): Suppose $\sum a_n = A$ and $\sum b_n = B$ with $\sum a_n$ converging absolutely. Let $\sum c_n$ be the Cauchy product. Then $\sum c_n = A \cdot B$

Pf: Set $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, $C_n = \sum_{k=0}^{n} c_k$, $\beta_n = B_n - B$
Then

$$C_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + \cdots + a_n b_0)$$

$$= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0$$

$$= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \cdots + a_n (B + \beta_0)$$

$$= A_n B + \underbrace{a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0}_{\text{call this } \gamma_n}$$

Since $A_n B \to AB$, suffices to show $\gamma_n \to 0$
Set $\alpha = \sum |a_n|$ and let $\varepsilon > 0$.
Since $\beta_n \to 0$ we can pick N with $\forall n \geq N$ $|\beta_n| < \varepsilon$
So for $n \geq N$

$$|\gamma_n - 0| = |\gamma_n| \leq |a_0||\beta_n| + |a_1||\beta_{n-1}| + \cdots |a_{n-N}||\beta_N| + |a_{n-N+1}||\beta_{N-1}| + \cdots + |a_n||\beta_0|$$

$$< \varepsilon(|a_0| + |a_1| + \cdots + |a_{n-N}|) + |a_{n-N+1}||\beta_{N-1}| + \cdots |a_n||\beta_0|$$

$$\leq \varepsilon \alpha + |a_{n-N+1}||\beta_{N-1}| + \cdots |a_n||\beta_0|$$

For n large enough, $|a_{n-N+1}||\beta_{N-1}| + \cdots + |a_n||\beta_0|$ will be less than ε since it converges to 0
So for large enough n, $|\gamma_n - 0| < \varepsilon(\alpha + 1)$. Thus $\gamma_n \to 0$ $\square$

Thm 3.51 (Abel) If $\sum a_n, \sum b_n, \sum c_n$ converge to A, B, C, where $\sum c_n$ is the Cauchy product then $C = A \cdot B$

Pf: In 140B (p.175)

Defn: If (k_n) is a seq in $\mathbb{N}$ using each natural number precisely once, and if $\sum a_n$ is a series and we set $a_n' = a_{k_n}$ then $\sum a_n'$ is called a rearrangement of $\sum a_n$

Thm 3.55: If $\sum a_n$ converges absolutely then every rearrangement converges to the same value.

Pf: Let (k_n) be seq in $\mathbb{N}$ using each natural number precisely once.
Let $\varepsilon > 0$ and pick N with $\forall m \geq n \geq N$ $\sum_{i=n}^{m} |a_i| < \varepsilon$

Now choose p with $\{k_0, k_1, \dots, k_p\} \supseteq \{0, 1, 2, \dots, N\}$

Then for $n > \max(p, N)$ we have

$$\left| \sum_{i=0}^{n} a_{k_i} - \sum_{i=0}^{n} a_i \right| < \varepsilon$$

$$\|$$

$$\left| \sum_{i \in \{k_0, \dots, k_n\} \setminus \{0, 1, \dots, n\}} a_i - \sum_{i \in \{0, 1, \dots, n\} \setminus \{k_0, \dots, k_n\}} a_i \right|$$

$\{k_0, \dots, k_n\} \setminus \{0, 1, \dots, n\} \subseteq \{i : i > N\}$ $\{0, 1, \dots, n\} \setminus \{k_0, \dots, k_n\} \subseteq \{i : i > N\}$

more explanation on next page $\square$

To write what I verbally said in lecture:

Pick $m \geq n$ with $\left(\{k_0, \dots, k_n\} \setminus \{0, \dots, n\}\right) \cup \left(\{0, \dots, n\} \setminus \{k_0, \dots, k_n\}\right) \subseteq \{N+1, \dots, m\}$

Then $\left| \sum_{i=0}^{n} a_{k_i} - \sum_{i=0}^{n} a_i \right| \leq \sum_{i=N+1}^{m} |a_i| < \varepsilon$

Lecture 20 Nov 20

HW 7 due today

2nd Midterm next week at two times:

- Class time 11:00-11:50 AM Wed Nov 25
- 12 hours later 11:00-11:50 PM Wed Nov 25

Thm 3.54 (Riemann): Suppose $\sum a_n$ converges non-absolutely and $-\infty \le \alpha \le \beta \le +\infty$. Then there is a rearrangement $\sum a_n'$ with partial sums s_n' satisfying

$$\liminf s_n' = \alpha, \quad \limsup s_n' = \beta$$

Pf: Set $p_n = \begin{cases} a_n & \text{if } a_n \ge 0 \\ 0 & \text{otherwise} \end{cases}$, $q_n = \begin{cases} -a_n & \text{if } a_n \le 0 \\ 0 & \text{otherwise} \end{cases}$

Note $p_n, q_n \ge 0$, $a_n = p_n - q_n$.
If $\sum p_n$ were to converge then $\sum q_n = \sum (p_n - a_n)$ would converge and $\sum |a_n| = \sum (p_n + q_n)$ would converge, contradiction. So $\sum p_n$ diverges. Similarly $\sum q_n$ diverges.

Let $P_1, P_2, \ldots$ be the non-negative terms from $a_1, a_2, \ldots$ (in order)
Let $Q_1, Q_2, \ldots$ be the absolute-value of the strictly negative terms from $a_1, a_2, \ldots$ (in order)
$\sum P_n$ differs from $\sum p_n$ only by 0 terms, so $\sum P_n$ diverge. Similarly $\sum Q_n$ diverges

Choose $\alpha_n, \beta_n \in \mathbb{R}$ with $\beta_1 > 0$, $\alpha_n < \beta_n$, $\alpha_{n-1} < \beta_n$, $\alpha_n \to \alpha$, $\beta_n \to \beta$

(Say $\beta_1 = |\beta| + 1$, $\beta_n = \beta + 2^{-n}$, $\alpha_n = \alpha - 2^{-n}$, when $\alpha, \beta \in \mathbb{R}$)

Let $m_1, k_1 \in \mathbb{Z}_+$ be least with

$$P_1 + P_2 + \cdots + P_{m_1} > \beta_1$$

$$P_1 + P_2 + \cdots + P_{m_1} - Q_1 - Q_2 - \cdots - Q_{k_1} < \alpha_1$$

Continue inductively, letting $m_n, k_n \in \mathbb{Z}_+$ be least with

$$x_n = P_1 + P_2 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + \cdots - Q_{k_{n-1}} + P_{m_{n-1}+1} + \cdots + P_{m_n} > \beta_n$$

$$y_n = P_1 + P_2 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + \cdots + P_{m_n} - Q_{k_{n-1}+1} - \cdots - Q_{k_n} < \alpha_n$$

Then $|x_n - \beta_n| < P_{m_n}$ and $|y_n - \alpha_n| < Q_{k_n}$

Since $\sum a_n$ converges, $P_n \to 0$, $Q_n \to 0$.

Since $\beta_n \to \beta$, $P_n \to 0$, we have $x_n \to \beta$

Since $\alpha_n \to \alpha$, $Q_n \to 0$, we have $y_n \to \alpha$

Thus α, β are least/greatest subseq. limits of the partial sums from

$P_1 + \cdots + P_{m_1} - Q_1 - \cdots \cdots \sim$

Further explanation below and next page

(*) (Partial sums are increasing from y_{n-1} to x_n and decreasing from x_n to y_n)

We will use Thm 3.17.
Let $\beta' > \beta$. Then there exists N
so that for all $n \geq N$ $\quad \beta_n + |P_n| < \beta'$ (since $\beta_n \to \beta$, $P_n \to 0$)

So if s' is a partial sum between y_{n-1} and x_n with $n \geq N$ then $m_n \geq n \geq N$

by (*) $\quad s' \leq x_n < \beta_n + P_{m_n} < \beta'$. Similarly when s' between x_n and y_n

So eventually all partial sums are strictly less than β'. So β is limsup of partial sums by Theorem 3.17. $\square$

Defn: Suppose (X, d_X), (Y, d_Y) are metric spaces, $E \subseteq X$, $f: E \to Y$, $p \in E'$. For a point $q \in Y$ we say the <u>limit of f at p is q</u> and write "$f(x) \to q$ as $x \to p$" or "$\lim_{x \to p} f(x) = q$" if:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in E \ \ 0 < d_X(x,p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon$$

Note: It may be that $p \notin E$ so $f(p)$ is not defined.
Even if $p \in E$ it can happen $f(p) \neq \lim_{x \to p} f(p)$

Thm 4.2: $\lim_{x\to p} f(x) = q$ iff for all seq's (p_n) in E

$$\left(\forall n\ p_n \neq p \text{ and } p_n \to p\right) \Rightarrow f(p_n) \to q$$

Pf: Assume $\lim_{x\to p} f(x) = q$. Let (p_n) be seq in E with $\forall n\ p_n \neq p$ and $p_n \to p$. Let $\varepsilon > 0$ and pick $\delta > 0$ with

$$\forall x \in E \quad 0 < d_X(x,p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon.$$

Since $p_n \to p$ there is N with

$$\forall n \geq N \ 0 < d_X(p_n, p) < \delta.$$

Then for $n \geq N$ we have $d_Y(f(p_n), q) < \varepsilon$.
Thus $f(p_n) \to q$

Now assume the statement "$\lim_{x\to p} f(x) = q$" is false.
Then there is $\varepsilon > 0$ so that

$$\forall \delta > 0 \ \exists x \in E \ \ 0 < d_X(x,p) < \delta \text{ and } d_Y(f(x), q) \geq \varepsilon.$$

For each $n \geq 1$, apply above with $\delta = \frac{1}{n}$ to obtain $p_n \in E$ satisfying

$$0 < d_X(p_n, p) < 1/n \text{ and } d_Y(f(p_n), q) \geq \varepsilon.$$

Then $p_n \to p$ and $\forall n\ p_n \neq p$ but $f(p_n) \not\to q$ $\square$

Cor: If f has a limit at p then the limit is unique

Lecture 21 Nov 23

Second Midterm on Wednesday

- Class time 11:00-11:50 AM Wed Nov 25
- 12 hours later 11:00-11:50 PM Wed Nov 25

My office hours this week: Tu 12:30-2:00 PM, 7:00-8:30 PM
The TA will have extra office hour W 8:00-9:00 AM

Defn: If $f,g: E \to \mathbb{C}$ then we obtain new functions

- $(f+g)(x) = f(x)+g(x)$ • $(f-g)(x) = f(x)-g(x)$
- $(fg)(x) = f(x)g(x)$ • $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (when $g(x) \neq 0$)

If $f,g: E \to \mathbb{R}$ we write $f \leq g$ if $\forall x \in E$ $f(x) \leq g(x)$.
Similarly if $\vec{f},\vec{g}: E \to \mathbb{R}^k$ we define

- $(\vec{f}+\vec{g})(x) = \vec{f}(x)+\vec{g}(x)$
- $(\vec{f}\cdot\vec{g})(x) = \vec{f}(x)\cdot\vec{g}(x)$
- for $\lambda \in \mathbb{R}$ $(\lambda\vec{f})(x) = \lambda\vec{f}(x)$

Thm 4.4: Let (X,d) be a metric space, $E \subseteq X$, $f,g: E \to \mathbb{C}$ and $p \in E'$. If $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$ then

- $\lim_{x\to p} (f+g)(x) = A+B$
- $\lim_{x\to p} (fg)(x) = AB$
- $\lim_{x\to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ if $B \neq 0$

Similarly, if $\vec{f},\vec{g}: E \to \mathbb{R}^k$, $\lim_{x\to p} \vec{f}(x) = \vec{A}$, $\lim_{x\to p} \vec{g}(x) = \vec{B}$
then
- $\lim_{x\to p} (\vec{f}+\vec{g})(x) = \vec{A}+\vec{B}$
- $\lim_{x\to p} (\vec{f}\cdot\vec{g})(x) = \vec{A}\cdot\vec{B}$

Pf: Follows from Theorem 3.3 and the previous theorem $\square$

Review for Second Midterm

Compact sets
Perfect sets
Connected sets

Convergence of sequences
Cauchy sequences, Cauchy criterion
Subsequences
liminf / limsup
Special sequences

e

Convergence of series

- Ratio test
- Root test
- Comparison

$\left.\right\}$ <u>test for absolute convergence</u>

- Alternating series test
- Summation by parts
- Cauchy criterion for series

Absolute convergence
Radius of convergence

1. Let (X,d) be a metric space, $(p_n)_{n\in\mathbb{N}}$ a seq. in X, and let $K\subseteq X$ be compact.

Prove that if no subseq. of (p_n) has limit point in K then there exists open set $U\supseteq K$ with $\{n\in\mathbb{N}: p_n\in U\}$ is finite

Pf: For each $q\in K$, q is not a subseq. limit of (p_n) so by theorem in class (not in book) there is $r(q)>0$ such that

$$\{n\in\mathbb{N}: p_n\in B_{r(q)}(q)\}\text{ is finite.}$$

The sets $B_{r(q)}(q)$, $q\in K$, are open and cover K. Since K is compact, there are $q_1, q_2, \dots, q_m\in K$ with $K\subseteq\bigcup_{i=1}^{m} B_{r(q_i)}(q_i)$.

Set $U=\bigcup_{i=1}^{m} B_{r(q_i)}(q_i)$. Then U is open and $U\supseteq K$. Finally,

$$\{n\in\mathbb{N}: p_n\in U\}=\bigcup_{i=1}^{m}\{n\in\mathbb{N}: p_n\in B_{r(q_i)}(q_i)\}$$

which is finite $\square$

2. Let $(a_n)_{n\in\mathbb{N}}$ be seq. of positive real numbers such that $\sum_{n\in\mathbb{N}} a_n$ converges. Prove that

$\sum_{n\in\mathbb{N}} 2a_n^2$ converges.

Pf: Since $\sum a_n$ converges, we have $a_n \to 0$. So there is N with $\forall n \geq N$ $|a_n - 0| < 1/2$, hence $\forall n \geq N$ $0 \leq a_n < 1/2$. It follows that $2a_n^2 < a_n$ for all $n \geq N$. Therefore $\sum 2a_n^2$ converges by comparison (for $n \geq N$) with $\sum a_n$ $\square$

3. Suppose $(s_n)_{n\in\mathbb{N}}$, $(t_n)_{n\in\mathbb{N}}$ are seq's of real numbers with $\forall n\in\mathbb{N}$ $s_n \leq t_n$. Prove

$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$$

(This is Theorem 3.19)

Pf: This is trivial if $\limsup_{n\to\infty} t_n = +\infty$ or $\limsup_{n\to\infty} s_n = -\infty$. So assume $\limsup_{n\to\infty} t_n \neq +\infty$ and $\limsup_{n\to\infty} s_n \neq -\infty$.
By Theorem 3.17(a) there is subseq (s_{n_i}) with $s_{n_i} \to \limsup_{n\to\infty} s_n$. Consider any $y \in \mathbb{R}$ with $y > \limsup_{n\to\infty} t_n$.
By Theorem 3.17(b) there is N with $\forall n \geq N$ $t_n < y$.
Pick m with $\forall i \geq m$ $n_i \geq N$. Then for all $i \geq m$ we have $s_{n_i} \leq t_{n_i} < y$, meaning $s_{n_i} \in (-\infty, y]$. By Theorem 3.3(5) (lecture-only) $\limsup_{n\to\infty} s_n = \lim_{i\to\infty} s_{n_i} \in (-\infty, y]$, so $\limsup_{n\to\infty} s_n \leq y$.
Since $y > \limsup_{n\to\infty} t_n$ was arbitrary, we conclude $\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$. $\square$

Lecture 22 Nov 30

HW 8 due Friday

Recall: If $f: E \to Y$ where $E \subseteq X$ and (X, d_X) and (Y, d_Y) are metric spaces, then for $p \in E'$ and $q \in Y$ the statement $\lim_{x \to p} f(x) = q$ means

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in E \quad 0 < d_X(x,p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon$$

Defn: Let (X, d_X), (Y, d_Y) be metric spaces, $E \subseteq X$, and $f: E \to Y$. We say f is continuous at $p \in E$ if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in E \quad d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$$

If f is continuous at every $p \in E$ then we say f is continuous on E (or continuous).

Thm 4.6: If $p \in E \setminus E'$ then every function $f: E \to Y$ is continuous at p. If $p \in E \cap E'$ then $f: E \to Y$ is continuous at p iff $\lim_{x \to p} f(x) = f(p)$.

Pf: If $p \in E \setminus E'$ then there is $\delta > 0$ with $B_\delta(p) \cap E = \{p\}$. So for all $x \in E$

$$d_X(x,p) < \delta \Rightarrow x = p \Rightarrow f(x) = f(p) \Rightarrow d_Y(f(x), f(p)) = 0.$$

So this δ works for all $\varepsilon > 0$.
The second statement is immediate from definitions. $\square$

Thm 4.7: Suppose (X, d_X), (Y, d_Y), (Z, d_Z) are metric spaces, $E_X \subseteq X$, $E_Y \subseteq Y$, $f: E_X \to E_Y$, $g: E_Y \to Z$. Define $h: E \to Z$ by $h(p) = g(f(p))$. If f is continuous at p and g is continuous at $f(p)$ then h is continuous at p.

Pf: Let $\varepsilon > 0$. Since g is cont. at $f(p)$, there is $r > 0$ with

$$\forall y \in E_Y \quad d_Y(y, f(p)) < r \Rightarrow d_Z(g(y), g(f(p))) < \varepsilon.$$

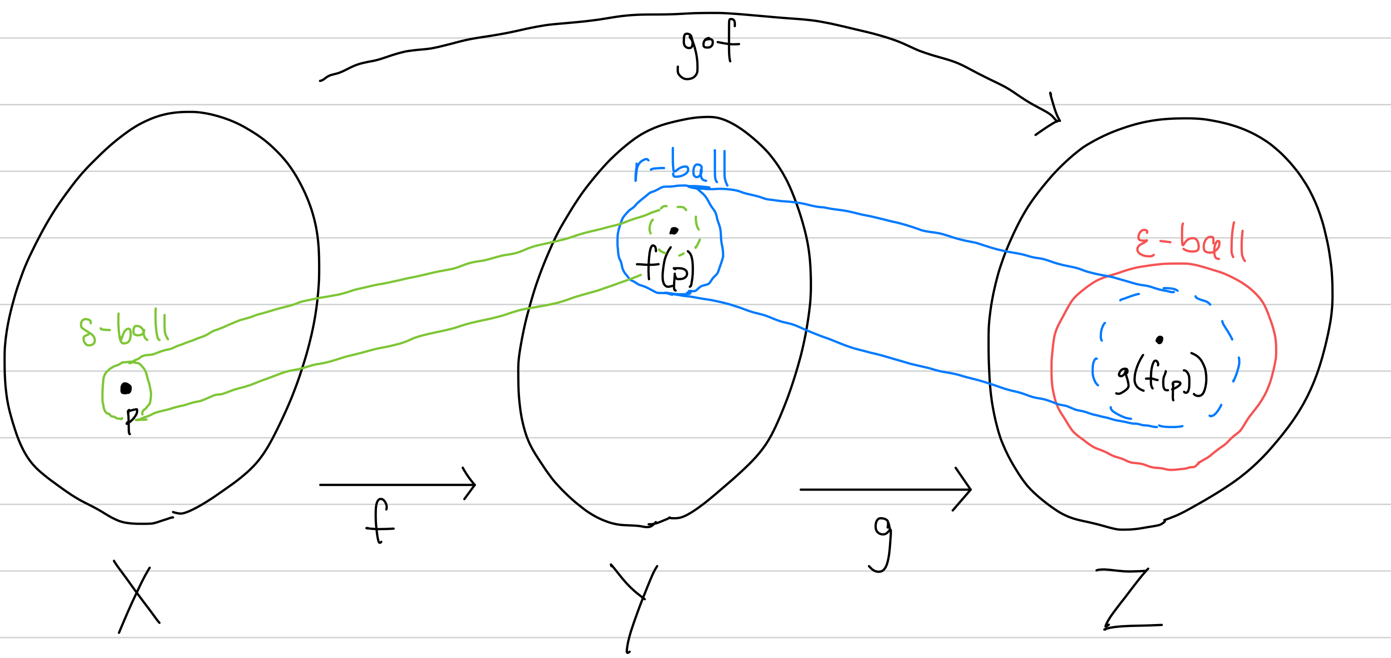
Since f is cont. at p, there is $\delta > 0$ with

$$\forall x \in E_X \quad d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < r.$$

It follows that

$$\forall x \in E_X \quad d_X(x, p) < \delta \Rightarrow d_Z\Big(\overset{h(x)}{\overset{\parallel}{g(f(x))}}, \overset{h(p)}{\overset{\parallel}{g(f(p))}}\Big) < \varepsilon.$$

We conclude h is cont. at p. $\square$



Note: The property of continuity of $f: E \to Y$ does not depend in any way on $X \setminus E$. It is therefore convenient to take the domain of f as the entire metric space.

Thm 4.8: $f: X \to Y$ is continuous (on X) iff $f^{-1}(V)$ is open for every open set $V \subseteq Y$.

Pf: Assume f is cont and let $V \subseteq Y$ be open. Let $p \in f^{-1}(V)$. Then $f(p) \in V$. Since V is open there is $\varepsilon > 0$ with $B_\varepsilon(f(p)) \subseteq V$, meaning

$$\forall y \in Y \quad d_Y(y, f(p)) < \varepsilon \Rightarrow y \in V.$$

Since f cont, there is $\delta > 0$ with

$$\forall x \in X \quad d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon.$$

It follows that $f(B_\delta(p)) \subseteq V$, meaning $B_\delta(p) \subseteq f^{-1}(V)$. Thus $f^{-1}(V)$ is open.

Now assume $f^{-1}(V)$ is open for all open $V \subseteq Y$. Fix $p \in X$ and let $\varepsilon > 0$. Set $V = B_\varepsilon(f(p))$. Then V is open so $f^{-1}(V)$ is open. Since $p \in f^{-1}(V)$ there is $\delta > 0$ with $B_\delta(p) \subseteq f^{-1}(V)$. So if $x \in X$ satisfies $d_X(x,p) < \delta$ then $x \in B_\delta(p) \subseteq f^{-1}(V)$ so $f(x) \in V = B_\varepsilon(f(p))$ and thus $d_Y(f(x), f(p)) < \varepsilon$.

We conclude f is continuous. $\square$

Cor: $f: X \to Y$ is cont. (on X) iff $f^{-1}(C)$ is closed for all closed sets $C \subseteq Y$.

Pf Sketch: This follows from previous theorem together with duality between open and closed sets and the fact that for all sets $D \subseteq Y$

$$f^{-1}(Y \setminus D) = X \setminus f^{-1}(D). \qquad \square$$

Thm 4.9: If $f, g: X \to \mathbb{C}$ are continuous then so are $f+g$, $f \cdot g$, $\frac{f}{g}$ (if $\forall x \in X$ $g(x) \neq 0$)

Pf: At isolated points there is nothing to prove. At limit points this follows from Theorem 4.4 and 4.6 $\square$

Thm 4.10: (A) Let $f_1, f_2, \dots, f_k : X \to \mathbb{R}$ and define $\vec{f}: X \to \mathbb{R}^k$ by $\vec{f}(x) = (f_1(x), \dots, f_k(x))$. Then $\vec{f}$ is cont. $\iff$ $\forall 1 \le i \le k$ f_i is cont.

(B) If $\vec{f}, \vec{g}: X \to \mathbb{R}^k$ are cont. then so are $\vec{f} + \vec{g}$ and $\vec{f} \cdot \vec{g}$

Pf: (A) Follows from Theorems 3.4, 4.2, 4.6

(B) Follows from Theorems 4.4 and 4.6 $\square$

Lecture 23 Dec 2

HW 8 due Friday

Obs: For $1 \le i \le k$ the map from $\mathbb{R}^k$ to $\mathbb{R}$ given by $\vec{x} = (x_1, \dots, x_k) \mapsto x_i$ is continuous (easy to check). Thus for $n_1, n_2, \dots, n_k \in \mathbb{N}$

$$(x_1, x_2, \dots, x_k) \mapsto x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

is continuous. So polynomials $P(\vec{x}) = \sum c_{n_1, n_2, \dots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ (where $c_{n_1, n_2, \dots, n_k} \in \mathbb{C}$ are fixed and all but finitely many are 0) are continuous. Additionally, rational functions $\frac{P(\vec{x})}{Q(\vec{x})}$ (P, Q are polynomials) are continuous on their domain. Also, similar to HW 2 Ch. 1 Prob 13 one can show that $||\vec{x}| - |\vec{y}|| \le |\vec{x} - \vec{y}|$, and it follows from this that the map $\vec{x} \in \mathbb{R}^k \mapsto |\vec{x}|$ is continuous.

Defn: A function $f: X \to Y$ is bounded if there is $q \in Y$ and $M > 0$ with $f(X) \subseteq B_M(q)$.

Thm 4.14: Let (X, d_X), (Y, d_Y) be metric spaces. If $f: X \to Y$ is continuous and X is compact then $f(X)$ is compact.

Pf: Let $\{V_\alpha : \alpha \in A\}$ be an open cover of $f(X)$. Since f is continuous, by Thm. 4.8 each $f^{-1}(V_\alpha)$ is open and $X = \bigcup_{\alpha \in A} f^{-1}(V_\alpha)$. Since X is compact, there are $\alpha_1, \dots, \alpha_n$ with $X = \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$. Then we have

$$f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\right) = \bigcup_{i=1}^{n} f(f^{-1}(V_{\alpha_i})) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$$

We conclude $f(X)$ is compact. $\square$

Thm 4.15: If $f: X \to \mathbb{R}^k$ is continuous and X is compact then $f(X)$ is closed and bounded.

Pf: Follows from previous theorem and Heine-Borel Thm 2.41 $\square$

Thm 4.16: Suppose (X, d_X) is compact metric space and $f: X \to \mathbb{R}$ is continuous. Set
$M = \sup_{x \in X} f(x) = \sup f(X)$, $m = \inf_{x \in X} f(x) = \inf f(X)$.
Then there are $p, q \in X$ with $f(p) = M$, $f(q) = m$.

Pf: $f(X)$ is closed and bounded by previous theorem, so $M, m \in f(X)$ $\square$

Obs: This means that f achieves its maximum/minimum.

Thm. 4.17: Let (X, d_X), (Y, d_Y) be metric spaces and let $f: X \to Y$. If X is compact and if f is a continuous bijection then $f^{-1}: Y \to X$ is continuous.

Pf: Since $(f^{-1})^{-1} = f$, the corollary to Theorem 4.8 tells us that

f^{-1} is continuous $\iff$ $f(C)$ is closed for all closed sets $C \subseteq X$.

Let $C \subseteq X$ be closed. Then C is compact, so by Thm 4.14 $f(C)$ is compact, hence $f(C)$ is closed. Thus f^{-1} is continuous. $\square$

Defn: Let (X, d_X), (Y, d_Y) be metric spaces and let $f: X \to Y$. We say f is *uniformly continuous* if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x_1, x_2 \in X \quad d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon$$

Obs: f being continuous means that if we fix $\varepsilon > 0$ then for every $x_1 \in X$ we can find δ_{x_1} (depending on x_1) with

$$\forall x_2 \in X \quad d_X(x_1, x_2) < \delta_{x_1} \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon$$

But with only continuity it may be that $\inf\{\delta_{x_1} : x_1 \in X\} = 0$. Uniform continuity means there is a fixed $\delta > 0$ that works for all $x_1 \in X$ simultaneously.

Thm 4.19: Let (X, d_X), (Y, d_Y) be metric spaces and let $f: X \to Y$. If f is continuous and X is compact then f is uniformly continuous.

Pf: Let $\varepsilon > 0$. Since f is continuous, for each $p \in X$ we can pick $\delta_p > 0$ with

$$\forall q \in X \quad d_X(p, q) < \delta_p \Rightarrow d_Y(f(p), f(q)) < \varepsilon/2$$

Set $V_p = B_{\frac{1}{2}\delta_p}(p)$.

Claim: If $q \in V_p$, $x \in X$ and $d_X(x,q) < \frac{1}{2}\delta_p$ then $d_Y(f(x), f(q)) < \varepsilon$

Pf of Claim: Since $q \in V_p$ (meaning $d_X(p,q) < \frac{1}{2}\delta_p$) and $d_X(x,q) < \frac{1}{2}\delta_p$, we have $d_X(p,x) < \delta_p$ (by triangle inequality) and $d_X(p,q) < \frac{1}{2}\delta_p < \delta_p$. So

$$d_Y(f(p), f(q)) < \varepsilon/2$$

$$d_Y(f(p), f(x)) < \varepsilon/2$$

and by triangle inequality $d_Y(f(q), f(x)) < \varepsilon$ $\square$ (Claim)

$\{V_p : p \in X\}$ is an open cover of X, so by compactness there are $p_1, \dots, p_n$ with $X = \bigcup_{i=1}^{n} V_{p_i}$.

Set $\delta = \frac{1}{2}\min(\delta_{p_1}, \dots, \delta_{p_n})$.

Consider $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$.

Since $X = \bigcup_{i=1}^{n} V_{p_i}$, there is $1 \le i \le n$ with $x_1 \in V_{p_i}$. Now Claim implies

$$d_Y(f(x_1), f(x_2)) < \varepsilon.$$

$\square$

Lecture 24 Dec 4

HW 8 due today

Email me by tomorrow if you can't take Final Exam at these times:

- Tuesday Dec 15 11:30 AM - 2:30 PM
- Tuesday Dec 15 11:30 PM - Wednesday Dec 16 2:30 AM

Thm 4.20: Let $E \subseteq \mathbb{R}$ be non-compact. Then

(A) $\exists f: E \to \mathbb{R}$ f is cont. but not bounded

(B) $\exists f: E \to \mathbb{R}$ f is cont. and bounded but has no maximum

If additionally E is bounded then

(C) $\exists f: E \to \mathbb{R}$ f is cont. but not uniformly cont.

Pf: Assume E is bounded. By Heine-Borel theorem E is not closed so there is $x_0 \in E' \setminus E$.

For (A) and (C) set $f(x) = \frac{1}{x-x_0}$ for $x \in E$.

Claim: f is not bounded (A)

Let $M > 0$. Since $x_0 \in E'$ we can find $x \in E$ with $|x_0 - x| < \frac{1}{M}$ For this x we have

$$|f(x)| = \frac{1}{|x-x_0|} > M.$$

Thus f is not bounded

Claim: f is not uniformly cont. (C)

Let $\varepsilon, \delta > 0$. First pick any p s.t. $p \in E$ and $|p - x_0| < \delta/2$. Since f is not bounded on $(x_0 - \delta/2, x_0 + \delta/2) \cap E$, we can find $q \in E$ with $|q - x_0| < \delta/2$ and $|f(q)| > |f(p)| + \varepsilon$. Then

$$|p-q| \leq |p - x_0| + |x_0 - q| < \delta \quad \text{but}$$

$$|f(q) - f(p)| \geq |f(q)| - |f(p)| > \varepsilon.$$ Thus f not unif. cont.

For (B) set $g(x) = \frac{1}{1+(x-x_0)^2}$

Claim: g is bounded and $\forall x \in E$ $g(x) < 1$ but $\sup_{x \in E} g(x) = 1$. (B)

Clearly $\forall x \in E$ $0 < g(x) < 1$ and g is bounded. Let $\varepsilon > 0$. Pick $x \in E$ with

$$|x - x_0| < \sqrt{\frac{1}{1-\varepsilon} - 1}\,.$$

For this x

$$g(x) = \frac{1}{1+(x-x_0)^2} > \frac{1}{1+\sqrt{\frac{1}{1-\varepsilon}-1}^{\,2}} = 1-\varepsilon$$

Thus $\sup_{x \in E} g(x) = 1$.

Now assume E is not bounded.
For (A) set $h(x) = x$ for $x \in E$
For (B) set $s(x) = \frac{x^2}{1+x^2}$ for $x \in E$

Claim: s is bounded and $\forall x \in E$ $s(x) < 1$ and $\sup_{x \in E} s(x) = 1$

It's clear that $\forall x \in E$ $0 \le s(x) < 1$ and s is bounded. Let $\varepsilon > 0$. Pick $x \in E$ s.t.

$$|x| > \sqrt{\frac{1}{\frac{1}{1-\varepsilon} - 1}}$$

For this x,

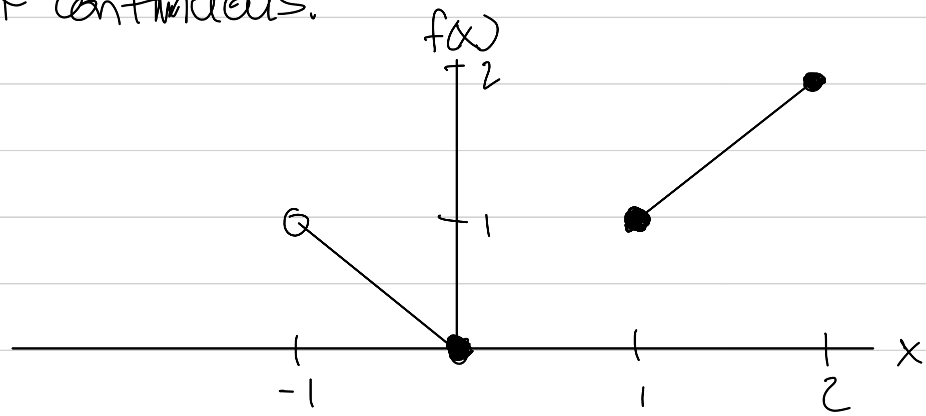
$$s(x) = \frac{x^2}{1+x^2} = \left(\frac{1}{x^2} + 1\right)^{-1} > 1-\varepsilon$$

Thus $\sup_{x \in E} s(x) = 1$. $\square$

Obs: ⓒ is not true if boundedness is not assumed.

Ex: $\mathbb{Z}$ is non-compact but every function $f:\mathbb{Z}\to\mathbb{R}$ is uniformly continuous.

Ex: Define $f:(-1,0]\cup[1,2]\to[0,2]$ by $f(x)=|x|$. Then f is a continuous bijection but f^{-1} is not continuous.



Thm 4.22: Let (X,d_X), (Y,d_Y) be metric spaces, and $f:X\to Y$. If $E\subseteq X$ is connected and f is continuous then $f(E)$ is connected.

Pf: Suppose $f(E)$ is not connected. Say $A,B\subseteq Y$ are nonempty separated and $A\cup B=f(E)$.

Set $G=f^{-1}(A)\cap E$, $H=f^{-1}(B)\cap E$. Then $E=G\cup H$ and G,H are nonempty.

Since $A\subseteq\bar{A}$, we have $G\subseteq f^{-1}(\bar{A})$. Since f is cont., $f^{-1}(\bar{A})$ is closed so $\bar{G}\subseteq f^{-1}(\bar{A})$. Therefore

$$\bar{G}\cap H\subseteq f^{-1}(\bar{A})\cap f^{-1}(B)=f^{-1}(\bar{A}\cap B)=f^{-1}(\emptyset)=\emptyset$$

So $\bar{G}\cap H=\emptyset$. Similarly, $G\cap\bar{H}=\emptyset$.

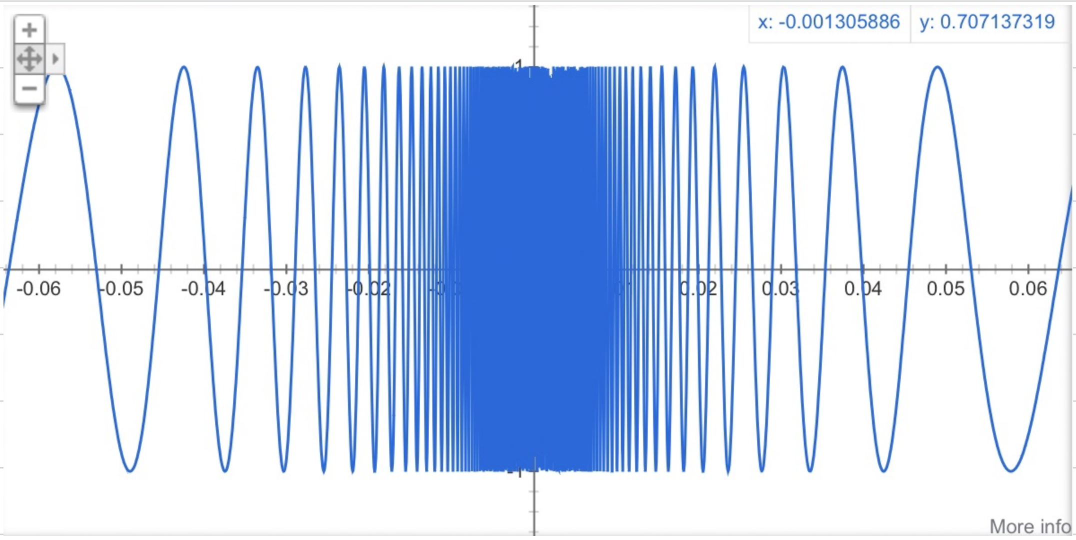
Thus G,H are separated and E is not connected. $\square$

Thm 4.23 (Intermediate Value Theorem): Let $f:[a,b]\to\mathbb{R}$ be continuous. If $f(a)<f(b)$ (or $f(b)<f(a)$) and $c\in\mathbb{R}$ satisfies $f(a)<c<f(b)$ (or $f(b)<c<f(a)$) then there is $x\in(a,b)$ with $f(x)=c$.

Pf: $[a,b]$ is connected (Thm. 2.47) and so by previous theorem $f([a,b])$ is connected. Since $f(a), f(b)\in f([a,b])$ and $f(a)<c<f(b)$, Theorem 2.47 implies that $c\in f([a,b])$. $\square$

Obs: Converse is false.

Ex: $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=\begin{cases}\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0\\ 0 & \text{otherwise}\end{cases}$



Lecture 25 Dec 7

HW 9 due Friday

Final Exam at two times:

- Tuesday Dec. 15 11:30 AM – 2:30 PM
- Tuesday Dec. 15 11:30 PM – Wednesday Dec. 16 2:30 AM

<u>Defn</u>: If f is not continuous at x and x is in the domain of f we say that f is <u>discontinuous at x</u>

<u>Defn</u>: Suppose f is a real-valued function defined on (a,b).

- For $a \leq x < b$ we write $f(x+) = q$ or $\lim_{t \to x^+} f(t) = q$ if
  $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall t \in \underbrace{(x, x+\delta)}_{\subseteq (a,b)} \; |f(t) - q| < \varepsilon$
- For $a < x \leq b$ we write $f(x-) = q$ or $\lim_{t \to x^-} f(t) = q$ if
  $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall t \in \underbrace{(x-\delta, x)}_{\subseteq (a,b)} \; |f(t) - q| < \varepsilon$

<u>Obs</u>: These definitions are equivalent to ones stated using limits of sequences just as in Theorem 4.2.

<u>Obs</u>: $\lim_{t \to x} f(t)$ exists iff $f(x+) = f(x-)$ and when this occurs $\lim_{t \to x} f(x)$ is equal to $f(x+) = f(x-)$

<u>Defn</u> If f is discontinuous at x and both $f(x+)$ and $f(x-)$ exist then we say f has a <u>discontinuity of the 1^{st} kind at x</u> or <u>simple discontinuity at x</u>. Otherwise, the discontinuity is said to be of the <u>2^{nd} kind</u>

both $f(x+)$ and $f(x-)$ exist and

Obs: Simple discontinuity if either $f(x+) \neq f(x-)$ or $f(x+) = f(x-)$ but $f(x) \neq f(x+) = f(x-)$

Ex:
- $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$
  has discontinuity of 2^{nd} kind at all points since $f(x-)$ and $f(x+)$ don't exist
- $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$
  f is cont. at $x=0$, discontinuity of 2^{nd} kind at all other points
- $f(x) = \begin{cases} x+2 & \text{if } -3 \leq x < -2 \\ -x-2 & \text{if } -2 \leq x < 0 \\ x+2 & \text{if } 0 \leq x \leq 1 \end{cases}$ $\quad f : [-3,1] \to \mathbb{R}$
  f is continuous on $[-3,1] \setminus \{0\}$
  simple discontinuity at 0

(assume we know what $\sin$ is and its properties)
- $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$
  f is continuous on $\mathbb{R} \setminus \{0\}$
  Discontinuity of 2^{nd} kind at 0

Defn: A function $f : (a,b) \to \mathbb{R}$ is <u>monotone increasing</u> if: whenever $a < x < y < b$ we have $f(x) \leq f(y)$.
Similarly f is <u>monotone decreasing</u> if whenever $a < x < y < b$ we have $f(x) \geq f(y)$

Thm 4.29 Let $f:(a,b)\to\mathbb{R}$ be monotone increasing. Then for every $x\in(a,b)$, $f(x-)$ and $f(x+)$ exist and

$$\sup_{a<t<x} f(t) = f(x-) \le f(x) \le f(x+) = \inf_{x<t<b} f(t).$$

Furthermore, if $a<x<y<b$ then $f(x+)\le f(y-)$.

Obs: Similar property holds when f monotone decreasing

Pf: $\{f(t): a<t<x\}$ is bounded above by $f(x)$.
So $A = \sup_{a<t<x} f(t)$ exists and $A\le f(x)$.
Fix $\varepsilon>0$. Since $A-\varepsilon$ is not an upperbound to $\{f(t): a<t<x\}$, there is $\delta>0$ with $A-\varepsilon < f(x-\delta)\le A$. So for any $t\in(x-\delta, x)$

$$A-\varepsilon < f(x-\delta) \le f(t) \le A \quad \text{so} \quad |f(t)-A|<\varepsilon.$$

Thus $f(x-)=A$. A similar argument shows $f(x+) = \inf_{x<t<b} f(t)$ and $f(x+)\ge f(x)$

Now suppose $a<x<y<b$. Pick any c with $x<c<y$. Then

$$f(x+) = \inf_{x<t<b} f(t) \le f(c) \le \sup_{a<t<y} f(t) = f(y-) \quad \square$$

Cor: Monotone functions have no discontinuities of the 2nd kind.

Thm 4.30 If f is monotone on (a,b) then it only has countably many discontinuities on (a,b).

Pf: Say f is increasing. Let E be set of discontinuities in (a,b). For each $x \in E$ pick $r(x) \in \mathbb{Q}$ satisfying $f(x-) < r(x) < f(x+)$. Then $r : E \to \mathbb{Q}$ is an injection because if $x_1 < x_2$ then by previous theorem

$$r(x_1) < f(x_1+) \leq f(x_2-) < r(x_2)$$

so $r(x_1) \neq r(x_2)$. Since $\mathbb{Q}$ is cntbl it follows E is cntbl $\square$

Ex: Given any cntbl set $E \subseteq (a,b)$ there is a monotone increasing function $f : (a,b) \to \mathbb{R}$ such that E is the set of discontinuities of f.

More explanation next time...

Lecture 26 Dec 9

HW 9 due Friday

Final Exam at two times:

- Tuesday Dec. 15 11:30 AM – 2:30 PM
- Tuesday Dec. 15 11:30 PM – Wednesday Dec. 16 2:30 AM

Ex: Given any countable set $E \subseteq (a,b)$ there is a monotone increasing function $f:(a,b)\to\mathbb{R}$ such that E is the set of discontinuities of f.

Say $E = \{e_1, e_2, e_3, \dots\}$. Fix seq (c_n) of positive real numbers with $\sum_{n=1}^{\infty} c_n$ convergent. Define for $x \in (a,b)$

$$I_x = \{n : e_n < x\}$$
$$I_x^+ = \{n : e_n \le x\}$$

Define $f(x) = \sum_{n\in I_x} c_n$ (this converges because $\sum_{n=1}^{\infty} c_n$ converges absolutely)

Then ① f is mono. increasing
② $f(e_n+) - f(e_n-) = c_n > 0$
③ f is cont. on $(a,b)\setminus E$

① holds since $x<t \Rightarrow I_x \subseteq I_t \Rightarrow f(x) \le f(t)$

For ② and ③ it suffices to show that for all $x\in(a,b)$

$$f(x-) = f(x) \text{ and } f(x+) = \sum_{n\in I_x^+} c_n$$

since then $f(e_k+) - f(e_k-) = \sum_{n\in I_{e_n}^+\setminus I_{e_n}} c_n = c_k$

and for $x\in(a,b)\setminus E$ we have $I_x = I_x^+$

and thus $f(x-) = f(x+)$ so f is cont. at x

We want to show that for $x \in (a,b)$
$f(x-) = f(x)$ and $f(x+) = \sum_{n \in I_x^+} c_n$

Note that when $t < x$
$[t, x) \cap \{e_1, e_2, \cdots, e_N\} = \emptyset \Rightarrow \forall 1 \le i \le N \ (e_i < t \Leftrightarrow e_i < x)$
$\Rightarrow I_x \setminus I_t \subseteq \{e_{N+1}, e_{N+2}, \cdots\}$
$\Rightarrow 0 \le f(x) - f(t) \le \sum_{n > N} c_n$

And when $x < t$
$(x, t) \cap \{e_1, e_2, \cdots, e_N\} = \emptyset \Rightarrow \forall 1 \le i \le N \ (e_i \le x \Leftrightarrow e_i < t.)$
$\Rightarrow I_t \setminus I_x^+ \subseteq \{e_{N+1}, e_{N+2}, \cdots\}$
$\Rightarrow 0 \le f(t) - \sum_{n \in I_x^+} c_n \le \sum_{n > N} c_n$

So given $\varepsilon > 0$ and $x \in (a,b)$ pick N with $\sum_{n > N} c_n < \varepsilon$ and choose $\delta > 0$ small enough so that $(x-\delta, x)$ and $(x, x+\delta)$ are disjoint with $\{e_1, e_2, \cdots, e_N\}$. Above observations show

$t \in (x-\delta, x) \Rightarrow |f(t) - f(x)| < \varepsilon$

$t \in (x, x+\delta) \Rightarrow |f(t) - \sum_{n \in I_x^+} c_n| < \varepsilon.$

Thus $f(x-) = f(x)$ and $f(x+) = \sum_{n \in I_x^+} c_n$.

Recall: A set $U \subseteq \mathbb{R}$ is a neighborhood of $x \in \mathbb{R}$ if U is open and $x \in U$.

Defn: A neighborhood of $+\infty$ is a set of the form $(M, +\infty)$, $M \in \mathbb{R}$.
A neighborhood of $-\infty$ is a set of the form $(-\infty, M)$, $M \in \mathbb{R}$

Defn: Let $E \subseteq \mathbb{R}$ and $f: E \to \mathbb{R}$. For $x, y \in \mathbb{R} \cup \{-\infty, +\infty\}$ we write $\lim_{t \to x} f(t) = y$ or $f(t) \to y$ as $t \to x$ if:

- either $x \in E'$
  or E is not bounded above and $x = +\infty$
  or E is not bounded below and $x = -\infty$
- and for every nbhd V of y there is a nbhd U of x such that

$$\forall t \in E \quad x \neq t \in U \Rightarrow f(t) \in V$$

Obs: When $x, y \in \mathbb{R}$ this notion coincides with the definition of limit that we learned before (start of Ch. 4)

Thm 4.34: Let $E \subseteq \mathbb{R}$ and let $f, g: E \to \mathbb{R}$.
Suppose $x, A, B \in \mathbb{R} \cup \{-\infty, +\infty\}$, $\lim_{t \to x} f(t) = A$, $\lim_{t \to x} g(t) = B$.
Then ① if $\lim_{t \to x} f(t) = A'$ then $A' = A$
② $\lim_{t \to x} (f+g)(t) = A + B$
③ $\lim_{t \to x} (fg)(t) = AB$
④ $\lim_{t \to x} \left(\frac{f}{g}\right)(t) = \frac{A}{B}$
provided the right-hand side is defined
($+\infty + (-\infty)$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{A}{0}$ are not defined)

Pf Sketch:

① Suppose towards contradiction $A \neq A'$. Say $A < A'$ (other case is similar). Then there is $r \in \mathbb{R}$ with $A < r < A'$. Then $V = (-\infty, r)$ and $V' = (r, +\infty)$ are nbhds of A and A' respectively. So there are nbhds U, U' of x such that

$$\forall t \in E \quad \begin{aligned} x \neq t \in U &\Rightarrow f(t) \in V \\ x \neq t \in U' &\Rightarrow f(t) \in V'. \end{aligned}$$

$U \cap U'$ is nbhd of x so we can find $t \in E$ with $x \neq t \in U \cap U'$. Then

$$f(t) \in V \cap V' = (-\infty, r) \cap (r, +\infty) = \emptyset,$$

a contradiction. So $A = A'$.

Rest are exercise. $\square$

Lecture 27 Dec 11

HW 9 due today

Final Exam at two times:

- Tuesday Dec. 15 11:30 AM – 2:30 PM
- Tuesday Dec. 15 11:30 PM – Wednesday Dec. 16 2:30 AM

Practice finals on Canvas page

Solutions to practice finals will be released later today

Office Hours next week:

- Vatsa: Mon. 9–11 AM
- Brandon: Mon. 8–10 PM

Review

Material from Ch. 4

- (Computing) limits of functions
- Verifying continuity / discontinuity
- Characterization of continuity via open/closed sets
- Connection between continuity and limits
- Continuity & connectivity, IVT
- Cont. images of compact sets are compact (closed, bounded in $\mathbb{R}^k$)
- Cont. functions achieve their max/min on compact sets
- Cont. bijections on compact sets have continuous inverses
- Cont. functions on compact sets are unif. cont.
- Verifying uniform continuity
- Types of discontinuities
- Discontinuities of monotone functions
- Neighborhoods of extended real numbers
- Limits at $\pm\infty$
- Limits with value $\pm\infty$

$f : X \to Y$

f continuous at p means

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall q \in X \ d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$

f continuous (on its domain) means

$\forall p \in X \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall q \in X \ d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$

f uniformly continuous means

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall p \in X \ \forall q \in X \ d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be unif. cont.
   (a) Prove $f+g$ is unif. cont.
   (b) Assume f, g are bounded. Prove fg is unif. cont.

2. Let $A \subseteq \mathbb{R}$ be nonempty, bounded above. Set $\alpha = \sup A$. Assume $\alpha \notin A$.
   Define $f : [0, +\infty) \to \mathbb{R}$ by $f(t) = \sup A \cap (-\infty, \alpha - t]$.
   Prove that $\lim_{t \to 0} f(t) = \alpha$.

3. Let $(x_n)_{n \in \mathbb{Z}_+}$ be seq. in $\mathbb{R}$ satisfying $x_{n+1} - x_n \geq \frac{1}{n}$.
   Prove $\lim_{n \to \infty} x_n = +\infty$.

4. Let $(s_n)_{n \in \mathbb{N}}$ be seq. in $\mathbb{R}$. Prove (s_n) has a subseq. limit in $\mathbb{R} \cup \{-\infty, +\infty\}$.

5. Let $(s_n)_{n \in \mathbb{N}}$ be bounded seq. in $\mathbb{R}$, and let $t \in \mathbb{R}$.
   Prove if the statement "$\lim_{n \to \infty} s_n = t$" is false
   then there is a convergent subseq. (s_{n_i})
   with $\lim_{i \to \infty} s_{n_i} \neq t$.

Assume $\alpha \notin A$

2. Let $A \subseteq \mathbb{R}$ be nonempty, bounded above. Set $\alpha = \sup A$.
Define $f : [0, +\infty) \to \mathbb{R}$ by $f(t) = \sup A \cap (-\infty, \alpha - t]$.
Prove that $\lim_{t \to 0} f(t) = \alpha$.

Pf: Let $\varepsilon > 0$. Since $\alpha - \varepsilon$ is not an upper bound to A, we can fix $a \in A$ with $a > \alpha - \varepsilon$. Since $\alpha \notin A$, we have $a \neq \alpha$. Then $\delta = \frac{\alpha - a}{2} > 0$.
Now suppose $t \geq 0$ and $|t - 0| < \delta$. Then $0 \leq t < \delta$ so

$$a \in A \cap (-\infty, \alpha - \delta] \subseteq A \cap (-\infty, \alpha - t]$$

so $f(t) = \sup A \cap (-\infty, \alpha - t] \geq a > \alpha - \varepsilon$.
Also $f(t) \leq \alpha$ since $A \cap (-\infty, \alpha - t] \subseteq A$. Thus
$|f(t) - \alpha| < \varepsilon$. We conclude $\lim_{t \to 0} f(t) = \alpha$. $\square$

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be unif. cont.
   (a) Prove $f+g$ is unif. cont.
   (b) Assume f, g are bounded. Prove fg is unif. cont.

Pf: (a). Let $\varepsilon > 0$. Since f, g are unif. cont, there are $\delta_f, \delta_g > 0$ such that for all $x, t \in \mathbb{R}$

$$|x-t| < \delta_f \Rightarrow |f(x) - f(t)| < \varepsilon/2$$
$$|x-t| < \delta_g \Rightarrow |g(x) - g(t)| < \varepsilon/2.$$

Set $\delta = \min(\delta_f, \delta_g) > 0$. If $x, t \in \mathbb{R}$ satisfy $|x-t| < \delta$ then $|x-t| < \delta_f$ and $|x-t| < \delta_g$ hence

$$\begin{aligned}
&|f(x) + g(x) - (f(t) + g(t))| \\
&\quad = |(f(x) - f(t)) + (g(x) - g(t))| \leq |f(x) - f(t)| + |g(x) - g(t)| \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus $f+g$ is unif. cont.

(b). Let $\varepsilon > 0$. Since f, g are bounded there are $M_f, M_g > 0$ such that

$$\forall x \in \mathbb{R} \quad |f(x)| \le M_f \,, \; |g(x)| \le M_g$$

Since f, g unif. cont, there are $\delta_f, \delta_g > 0$ such that

$$\forall x, t \in \mathbb{R} \quad \begin{array}{l} |x-t| < \delta_f \Rightarrow |f(x) - f(t)| < \frac{\varepsilon}{2M_g} \\ |x-t| < \delta_g \Rightarrow |g(x) - g(t)| < \frac{\varepsilon}{2M_f} . \end{array}$$

Set $\delta = \min(\delta_f, \delta_g) > 0$. Then for $x, t \in \mathbb{R}$ with $|x-t| < \delta$

we have

$$\begin{aligned} |f(x)g(x) - f(t)g(t)| &= |f(x)g(x) - f(t)g(x) + f(t)g(x) - f(t)g(t)| \\ &\le |f(x) - f(t)| \cdot |g(x)| + |f(t)| \cdot |g(x) - g(t)| \\ &\le M_g \, |f(x) - f(t)| + M_f \cdot |g(x) - g(t)| \\ &< M_g \cdot \frac{\varepsilon}{2M_g} + M_f \cdot \frac{\varepsilon}{2M_f} = \varepsilon \end{aligned}$$

Thus fg is unif. cont. $\square$