Math 140A

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Office Hours

How to construct $\mathbb{R}$ from $\mathbb{Q}$ (as in appendix)

Defn: A set $\alpha \subseteq \mathbb{Q}$ is a cut if:

1. $\emptyset \neq \alpha \neq \mathbb{Q}$
2. If $p \in \alpha$ and $q \in \mathbb{Q}$, $q < p$ then $q \in \alpha$
3. If $p \in \alpha$ then $\exists r \in \alpha$ $r > p$

Intuition: A cut is a set of the form

$$\alpha = (-\infty, x) \cap \mathbb{Q}$$

The construction will identify x with α

$\mathbb{R}$ will be defined as the set of all cuts.

Question related to Piazza post

Orignal Q:

If $b>1$ and $0<y<1$ does there exist real r with $b^r<y$

$$b^r<y \iff b^{-r}>\frac{1}{y}$$

Claim: If $b>1$ and $y\in\mathbb{R}$ then $\exists\, n\in\mathbb{N}$ $b^n>y$

Hint for proving claim: Modify proof of Thm. 1.20(a)

For 7(f):

Claim: If $x,y\in\mathbb{R}$ and $x<y$ then $b^x<b^y$ (for $b>1$)

This is proven in 6(c)

HW 2 # 6 (d)

If you change the definition to

$$B(x) = \{ b^t : t \in \mathbb{Q},\ t \le x \}$$

then you can show when $x, y \in \mathbb{R} \setminus \mathbb{Q}$ and $x+y \in \mathbb{Q}$

$\forall x, y \in \mathbb{R}$ $\quad B(x) \cdot B(y) \not\supseteq B(x+y)$

$$B(x)\cdot B(y) = \{ z \cdot w : z \in B(x),\ w \in B(y) \}$$

Want to prove $b^x \cdot b^y = b^{x+y}$,

meaning $(\sup B(x)) \cdot (\sup B(y)) = \sup B(x+y)$

Strategy: Check that $(\sup B(x)) \cdot (\sup B(y))$ satisfies the two conditions for being $\sup B(x+y)$

Check: $(\sup B(x)) \cdot (\sup B(y))$ is upper bound to $B(x+y)$. Hint: Use $B(x+y) \subseteq B(x) \cdot B(y)$

Check: Anything less than $(\sup B(x)) \cdot (\sup B(y))$ is not upper bound to $B(x+y)$.

— Continued —

Show $B(x+y) \subseteq B(x) \cdot B(y)$

Consider $b^t \in B(x+y)$. So $t \in \mathbb{Q}$, $t < x+y$

Want to find $b^{t_1} \in B(x)$, $b^{t_2} \in B(y)$, $t_1 + t_2 = t$

Want to find $t_1, t_2 \in \mathbb{Q}$, $t_1 + t_2 = t$, $t_1 < x$, $t_2 < y$

<u>Hint</u>: Since $t < x+y$ we have $\underline{t-y} < x$

think about this

HW 2 #7 (ab)

(a) Hint: Use the identity

$$b^n - a^n = (b-a)\underbrace{\left(b^{n-1} + ab^{n-2} + \cdots + a^{n-2}b + a^{n-1}\right)}_{n \text{ terms}}$$

When $b>1$ and $a=1$ we have $a^k b^{n-k-1} \geq 1$ $(k=0,1,\cdots,n-1)$

(b) Replace the b in (a) with $b^{1/n}$

Check: when $b>1$, $b^{1/n} > 1$ as well

$A, B \subseteq (0, +\infty)$ nonempty bounded above.

Prove $(\sup A)(\sup B) = \sup\{ab : a \in A, b \in B\}$

Pf: Set $\alpha = \sup A$, $\beta = \sup B$ (exist by lub property).

If $a \in A$ and $b \in B$ then $a \leq \alpha$ and $b \leq \beta$
so $ab \leq \alpha b \leq \alpha\beta$ (since $\alpha, \beta, a, b \geq 0$)
So $\alpha\beta$ is upperbound to $\sup\{ab : a \in A, b \in B\}$.

Notice $A \neq \emptyset$ so $\exists a \in A$ and $a > 0$, thus $\alpha \geq a > 0$
Similarly $\beta > 0$.

Now consider $x < \alpha\beta$.
Then $\frac{x}{\beta} < \alpha$ so there is $a \in A$ with $\frac{x}{\beta} < a$.
We have $\frac{x}{a} < \beta$ so there is $b \in B$ with $\frac{x}{a} < b$.
Hence $x < ab$.
So x is not an upperbound to $\{ab : a \in A, b \in B\}$
We conclude $\alpha\beta = \sup\{ab : a \in A, b \in B\}$

HW 3 Problem A

$X = \{ f : \mathbb{N} \to \mathbb{N} \mid f \text{ is injective} \}$

Let $F \subseteq X$ be countably infinite. We can write $F = \{ f_0, f_1, f_2, \cdots \}$. Inductively define $g : \mathbb{N} \to \mathbb{N}$ by setting $g(0) = f_0(0) + 1$ and once $g(0), \cdots, g(n-1)$ are defined, set

$$g(n) = \max(g(0), g(1), \cdots, g(n-1), f_n(n)) + 1.$$

It is clear from the induction that $\forall k < n$ $g(n) \neq g(k)$ (since $g(n) \geq g(k) + 1$). So g is an injection, hence $g \in X$. But for every $n \in \mathbb{N}$ we have $g(n) \geq f_n(n) + 1$, hence $g(n) \neq f_n(n)$ and $g \neq f_n$. So $g \in X \setminus F$. Thus $F \neq X$. We conclude X is uncountable. $\square$

<u>Sketch Easier proof</u>: Define $\phi : \{0,1\}^{\mathbb{N}} \to X$ by

$$\phi(f)(n) = 2n + f(n) \quad \text{for } f : \mathbb{N} \to \{0,1\} \text{ and } n \in \mathbb{N}.$$

<u>Check</u>: ϕ is an injection.

Then ϕ is bijective with its image $Y = \phi(\{0,1\}^{\mathbb{N}})$.
We learned $\{0,1\}^{\mathbb{N}}$ is uncountable, so Y is uncountable. Since $Y \subseteq X$, we conclude X is uncountable (Thm. 2.8) $\square$

<u>Notation</u>: Y^X is the set of all functions $f : X \to Y$

Practice Midterm B #3

Let $Y \subseteq X$ be countably infinite. Then we can write Y as $Y = \{A_0, A_1, A_2, \dots\}$.
Define $B \subseteq \mathbb{N}$ by declaring for each $n \in \mathbb{N}$:

$$(2n \in B \Leftrightarrow 2n \notin A_n) \text{ and } (2n+1 \in B \Leftrightarrow 2n+1 \notin A_n).$$

Since precisely one of $2n, 2n+1$ are in A_n but not both, we have that precisely one of $2n, 2n+1$ are in B but not both. This holds for every $n \in \mathbb{N}$ so $B \in X$. But for every $n \in \mathbb{N}$ we have $B \neq A_n$ since $2n$ is an element of precisely one of the sets B, A_n. Since $\forall n \in \mathbb{N}$ $B \neq A_n$, we have $B \notin Y$, meaning $B \in X \setminus Y$.
Thus $Y \neq X$. We conclude that X is uncountable. $\square$

Ch. 3 #8. Assume $\sum a_n$ converges, (b_n) mono. and bounded.
Prove $\sum a_n b_n$ converges

Pf 1: Since (b_n) is mono. & bounded, it converges to some $\beta \in \mathbb{R}$. Notice $\sum a_n b_n$ converges iff $\sum a_n(-b_n)$ converges. So by replacing (b_n) with $(-b_n)$ if necessary, we can assume (b_n) is mono. decreasing.
Set $b_n' = b_n - \beta$. Then $b_0' \geq b_1' \geq \cdots \geq 0$, $\lim_{n\to\infty} b_n' = 0$.

Since $\sum a_n$ converges, its partial sums are bounded.
By Thm 3.42, $\sum a_n b_n'$ converges.

Also $\sum a_n \beta$ converges to $\beta \cdot \sum a_n$. (Thm 3.47)
Since $a_n b_n = a_n b_n' + a_n \beta$, it follows that
$\sum a_n b_n$ converges to $\sum a_n b_n' + \beta \cdot \sum a_n$. $\square$

Pf 2: Set $A_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$, set $A_{-1} = 0$.

Since $\sum a_n$ converges, the partial sums A_n are bounded. So there is $M > 0$ with $\forall n \; |A_n| < M$.

Let $\varepsilon > 0$. Since (b_n) is mono. & bounded, it converges hence is Cauchy. So there is N_1 with $|b_n - b_m| < \varepsilon/3M$ for all $m \geq n \geq N_1$. Say $\lim_{n\to\infty} b_n = b$, and $\sum a_n = A = \lim_{n\to\infty} A_n$. By Theorem 3.3, $A_n b_n \to Ab$ and $A_{n-1} b_n \to Ab$. So there are N_2 with $|A_n b_n - Ab| < \varepsilon/3$ for all $n \geq N_2$ and N_3 with $|A_{n-1} b_n - Ab| < \varepsilon/3$ for all $n \geq N_3$.

So whenever $m \geq n \geq \max(N_1, N_2, N_3)$ we have

$$\left|\sum_{k=n}^{m} a_k b_k\right| = \left|\sum_{k=n}^{m-1} A_n (b_n - b_{n+1}) + A_m b_m - A_{n-1} b_n\right|$$

$$\leq \sum_{k=n}^{m-1} |A_n| \cdot |b_n - b_{n+1}| + |A_m b_m - A_{n-1} b_n|$$

$$< M \cdot \left(\sum_{k=n}^{m-1} |b_n - b_{n+1}|\right) + |A_m b_m - A_{n-1} b_n|$$

$$= M \cdot |b_n - b_m| + |A_m b_m - A_{n-1} b_n|$$

$$< \frac{\varepsilon}{3} + |A_m b_m - Ab| + |Ab - A_{n-1} b_n|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus $\sum a_n b_n$ converges by Cauchy criterion. $\square$

HW 7 Problem B.

$\sum a_n$ converges absolutely $\iff$ $\forall$ bounded (b_n) $\sum a_n b_n$ converges

Pf of $\Leftarrow$: Define

$$b_n = \begin{cases} 1 & \text{if } a_n \geq 0 \\ -1 & \text{if } a_n < 0 \end{cases}$$

Then $\forall n$ $a_n b_n = |a_n|$.

The seq. (b_n) is bounded so by assumption $\sum a_n b_n = \sum |a_n|$ converges. Thus $\sum a_n$ converges absolutely. $\square$

Wrong!

For all bounded seq's (b_n) prove that

$\sum a_n b_n$ converges $\Rightarrow \sum a_n$ converges absolutely.

Musings on series in other fields (not related to our course)

$$S = 1 + 2 + 4 + 8 + \cdots \in \mathbb{R}$$

$$2S + 1 = S$$

$$S = -1 \qquad \text{False}$$

$$S = 1 + 2 + 4 + 8 + \cdots \in F \qquad (F \text{ some field})$$

Supposing F has a "nice" notion of convergence and assuming the series converges then it is true that

$$2S + 1 = S$$

so $S = -1$

additive inverse of multiplicative identity in F.

Say $2^n = 0$ (in F). Then

$$S = 1 + 2 + 4 + 8 + \cdots + 2^{n-1} = 2^n - 1 = -1$$