The Devil's Staircase

Recall the usual construction of the Cantor set: $C_0 = [0,1]$, $C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$, and in general C_n is a disjoint union of 2^n closed intervals, each of length 3^{-n}, constructed from C_{n-1} by deleting the open-middle-third of each of the 2^{n-1} intervals constituting C_{n-1}. Then the total length of the intervals in C_n is $(\frac{2}{3})^n$. The Cantor set C is the intersection $C = \bigcap_{n\geq 0} C_n$.

Set $g_n = (\frac{3}{2})^n \mathbb{1}_{C_n}$; that is,

$$g_n(x) = \begin{cases} (\frac{3}{2})^n, & x \in C_n \\ 0, & x \notin C_n \end{cases}.$$

The function g_n is discontinuous only at the 2^{n+1} points at the boundaries of the intervals making up C_n. This is a finite set, and so g_n is Riemann integrable. So we may define $f_n\colon [0,1] \to \mathbb{R}$ as follows:

$$f_n(x) = \int_0^x g_n(t)\,dt.$$

From the Fundamental Theorem of Calculus, we know that the functions f_n are Lipschitz continuous. Note also that $f_n(0) = 0$, while $f_n(1) = \int_0^1 g_n(t)\,dt$. This integral can be calculated as

$$\int_0^1 g_n(t)\,dt = (\tfrac{3}{2})^n \int_0^1 \mathbb{1}_{C_n}(t)\,dt = (\tfrac{3}{2})^n \cdot \text{length}(C_n) = 1.$$

So f_n is a continuous function with $f_n(0) = 0$ and $f_n(1) = 1$ for each n. Notice that f_n is constant on $[0,1] - C_n$, and is linear with slope $(\frac{2}{3})^n$ on the intervals making up C_n. So f_n is monotone increasing. Here is the graph of f_2.

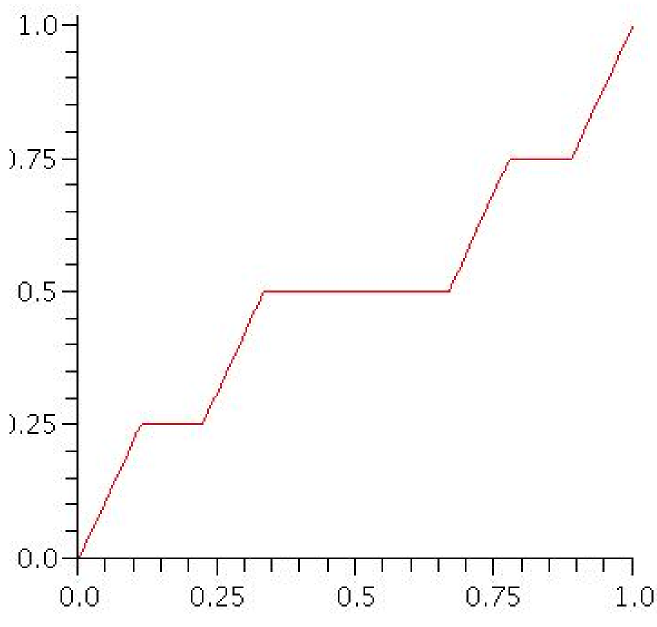

FIGURE 1. The graph of f_2.

Let I be any one of the 2^n closed intervals that make up C_n. Then $g_n(x) = (\frac{3}{2})^n$ for all $x \in I$, while $g_{n+1}(x)$ is equal to $(\frac{3}{2})^{n+1} = \frac{3}{2}g_n(x)$ for x in the first third or last third of I, and equal to 0 in the middle interval. It follows that

$$\int_I g_n(t)\,dt = \int_I g_{n+1}(t)\,dt = 2^{-n}. \tag{$*$}$$

Note also that $g_n(x) = g_{n+1}(x)$ for $x \notin C_n$. Hence, if a is any endpoint of an interval in C_n,

$$\int_0^a g_n(t)\,dt = \int_0^a g_{n+1}(t)\,dt. \tag{$**$}$$

It follows by integrating $(*)$ that

$$f_{n+1}(x) = f_n(x), \quad x \notin C_n. \tag{$***$}$$

Now, the Cantor set C is closed, and so any point $x \notin C$ is contained in a small open interval in $C^c = \bigcup_{n\geq 0} C_n^c$. Therefore there is some $N \in \mathbb{N}$ such that $x \in C_n^c$ for all $n \geq N$, and by the above we have $f_n(x) = f_N(x)$ for all $n \geq N$. This shows that the sequence of functions $f_n(x)$ converges pointwise on C^c. But it's even better than that.

Let $x \in C_n$, and let $I = [a, b]$ be the interval in C_n containing x. Applying $(**)$, we have

$$\begin{aligned} |f_n(x) - f_{n+1}(x)| = \left|\int_0^x [g_n(t) - g_{n+1}(t)]\,dt\right| &= \left|\int_a^x [g_n(t) - g_{n+1}(t)]\,dt\right| \\ &\leq \int_a^x |g_n(t) - g_{n+1}(t)|\,dt \\ &\leq \int_a^b |g_n(t) - g_{n+1}(t)|\,dt. \end{aligned}$$

Again, we have $|g_n(t) - g_{n+1}(t)| = (\frac{3}{2})^{n+1} - (\frac{3}{2})^n$ on the first and last third of $[a, b]$, while it equals $(\frac{3}{2})^n$ on the middle third. The difference is therefore bounded above by $(\frac{3}{2})^{n+1}$ on the interval which has length 3^{-n}, and so the above integral is bounded by $3^{-n}(\frac{3}{2})^{n+1}$; that is, we have proved that

$$|f_n(x) - f_{n+1}(x)| \leq \frac{3}{2} \cdot 2^{-n} < 2^{-n+1}, \quad x \in C_n.$$

Combining this with $(***)$, we have $|f_n(x) - f_{n+1}(x)| < 2^{-n+1}$ for all x. It follows that the sequence $\{f_n\}$ is uniformly Cauchy, and therefore converges uniformly to a limit function f. The functions f_n are continuous, and so the uniform limit function f is continuous. Also, for $x \notin C_n$ the sequence $\{f_k(x)\}_{k=n}^{\infty}$ is constant, and therefore $f_n(x) = f(x)$. But f_n is constant on C_n^c. Finally, since $f_n(0) = 0$ and $f_n(1) = 1$ for all n, we have $f(0) = 0$ and $f(1) = 1$. We have therefore proved the following:

$f_n \to f$ *uniformly, where* f *is continuous,* $f'(x) = 0$ *for* $x \in C^c$*, and* $f(0) = 0$ *and* $f(1) = 1$.

A little additional thought shows that the limit function f is monotone increasing. It is called the *Cantor Function* or *the Devil's staircase*. Its graph is shown in Figure 2. It can actually be described in simple terms. Here is an algorithm for calculating $f(x)$ for $x \in [0, 1]$.

- Express x in base 3, $[x]_3$. (Choose the representation that does not end in $1111\ldots$)
- If $[x]_3$ contains any 1s, with the first 1 being at position n: $[x]_3 = 0.x_1x_2\ldots x_{n-1}1x_{n+1}\ldots$, replace the number with $T(x)$ in ternary $[T(x)]_3 = 0.x_1x_2\ldots x_{n-1}2$. Otherwise, if $[x]_3$ contains no 1s (i.e. $x \in C$), then $T(x) = x$.

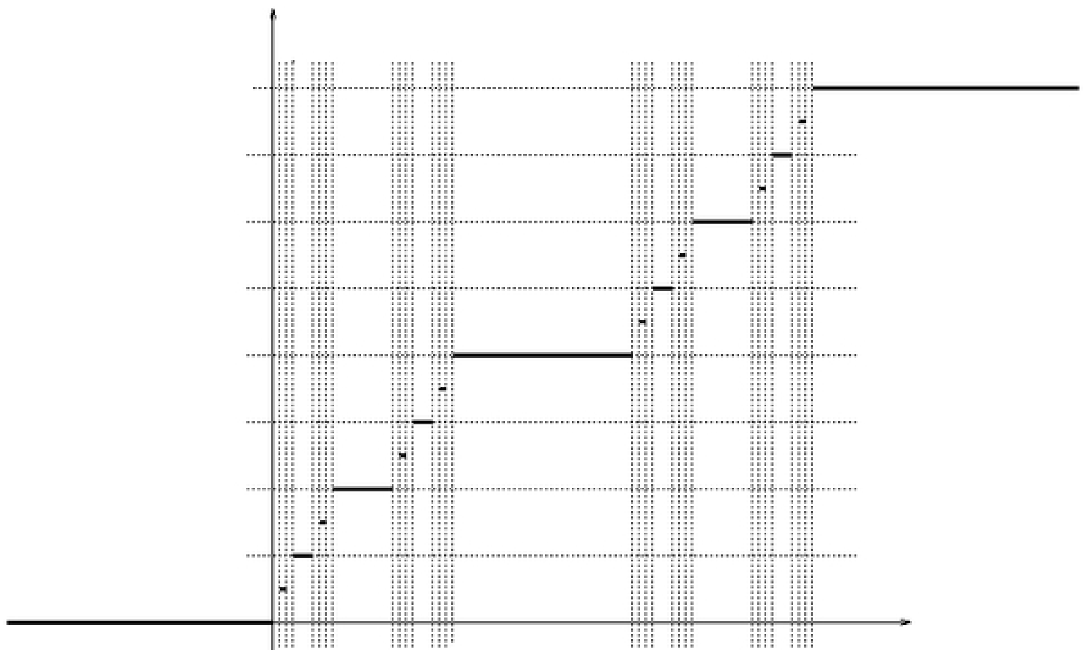

FIGURE 2. The graph of the Devil's staircase.

- Since $T(x)$ is an even ternary number (i.e. in C), we can relabel all 2s as 1s. The 0-1 string that remains, re-interpretted in base 2, is $f(x)$.

For example, $1/4$ in base 2 is $0.0202020\ldots$ (isn't that surprising? $1/4 \in C$!). Hence, $f[(1/4)]_2 = 0.01010101\ldots = 1/3$. On the other hand, $[1/5]_3 = 0.012101210\ldots$, hence $[T(1/5)]_3 = 0.02$, and so $[f(1/5)]_2 = 0.01$, which is $1/4$ in base 2; hence, $f(1/5) = 1/4$.

In general, a non-constant monotone increasing function that is continuous, differentiable almost everywhere with derivative 0, is called *singular*. These functions really point out the necessity of the continuity (and definition everywhere) of the derivative of f in order for the Fundamental Theorem of Calculus to make sense. (After all, the derivative of f is 0 almost everywhere, and so $\int f' \neq f$ in this case.)

For other fun examples of singular functions, lookup the Wikipedia entries on *Minkowski's question mark function*, and *Kolmogorov's circle map*.

One final note: for those who are physically inclined, the 1998 Nobel Prize in Physics was awarded for the discovery and explanation of what is known as the *fractional quantum Hall effect*. This phenomenon is a physical manifestation of a singular function.