1. Let $f : \mathbb{R} \to \mathbb{R}$, let $n \in \mathbb{N}$, and suppose that $f^{(n)}$ is defined on $\mathbb{R}$ and $f^{(n)}(x) = 0$ for all $x \in \mathbb{R}$. Prove that $f$ is a polynomial.

**Solution:** Set $P(x) = f(0) + f'(0) \cdot x + \cdots + f^{(n-1)}(0) \cdot x^{n-1}$. Then $P$ is a polynomial. By Taylor’s Theorem, for every $x \in \mathbb{R}$ there is $t \in \mathbb{R}$ with $f(x) = P(x) + \frac{f^{(n)}(t)}{n!} \cdot x^n$. Since $f^{(n)}(t) = 0$ for all $t$, it follows that $f(x) = P(x)$ for all $x \in \mathbb{R}$. Thus $f = P$. □
2. Let $f : [0, 1] \to \mathbb{R}$, let $n \in \mathbb{Z}_+$, and suppose that $f^{(n)}$ is defined on $[0, 1]$. Assume that $f(0) = f(\frac{1}{n}) = f(\frac{2}{n}) = \cdots = f(\frac{n-1}{n}) = f(1)$. Prove there is $x \in [0, 1]$ with $f^{(n)}(x) = 0$.

**Solution:** We will prove by induction that for every $0 \leq k \leq n$ there are $(n + 1 - k)$-many distinct points $0 \leq x_0 < x_1 < \cdots < x_{n-k} \leq 1$ such that $f^{(k)}(x_0) = f^{(k)}(x_1) = \cdots = f^{(k)}(x_{n-k})$, and moreover that all of these values are 0 when $k \geq 1$ (exercise caution with this statement - the $x_i$’s depend on $k$). By assumption, the base case $k = 0$ is satisfied with $x_i = \frac{i}{n}$.

Now inductively assume this property holds for $k$, where $0 \leq k < n$. Let $x_i (0 \leq i \leq n - k)$ be as described. Then for each $0 \leq i \leq n - k - 1$, since $f^{(k)}(x_i) = f^{(k)}(x_{i+1})$, the Mean Value Theorem (applied to $f^{(k)}$) provides a point $y_i \in (x_i, x_{i+1})$ with $f^{(k+1)}(y_i) = 0$. We therefore have $n + 1 - (k + 1) = n - k$ many distinct points $0 \leq y_0 < y_1 < \cdots < y_{n-k-1} \leq 1$ with $f^{(k+1)}(y_0) = \cdots = f^{(k+1)}(y_{n-k-1}) = 0$. This completes the inductive step. So our claim holds for all $0 \leq k \leq n$. Applying this property with $k = n \geq 1$, we see that there is $x \in [0, 1]$ with $f^{(n)}(x) = 0$. \qed
3. Let \( f : [0,1] \to \mathbb{R} \). Assume that \( f(x) \geq 0 \) for all \( x \in [0,1] \) and that for every \( \epsilon > 0 \) the set \( \{ x \in [0,1] : f(x) > \epsilon \} \) is finite. Prove that \( f \) is bounded, is integrable on \([0,1]\), and compute (with justification) the value of \( \int_0^1 f \, dx \).

**Solution:** Set \( D = \{ x \in [0,1] : f(x) > 1 \} \). Then \( D \) is finite and for all \( x \in [0,1] \) we have \( 0 \leq f(x) \leq M = \max(\{1\} \cup \{f(x) : x \in D\}) \). Thus \( f \) is bounded.

We claim that \( \int_0^1 f \, dx = 0 \). Since \( f(x) \geq 0 \) for all \( x \in [0,1] \), it is immediate that \( L(P,f) \geq 0 \) for every partition \( P \). So it suffices to show that for every \( \epsilon > 0 \) there is a partition \( P \) with \( U(P,f) < \epsilon \).

Fix \( \epsilon > 0 \). Set \( E = \{ x \in [0,1] : f(x) \geq \frac{\epsilon}{2} \} \). Our assumptions imply that \( E \) is finite. So we can find a partition \( P = \{x_0,x_1,\ldots,x_n\} \) of \([0,1]\) with the property that whenever \( e \in E \) and \( e \in [x_{i-1},x_i] \) we have \( \Delta x_i < \frac{\epsilon}{4M|E|} \). Set \( A = \{1 \leq i \leq n : M_i < \frac{\epsilon}{2}\} \) and \( B = \{1 \leq i \leq n : M_i \geq \frac{\epsilon}{2}\} \). Notice that due to overlapping of endpoints, there are at most two sub-intervals of \( P \) for each \( e \in E \), meaning \( |B| \leq 2|E| \). So we have

\[
U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i
= \sum_{i \in A} M_i \Delta x_i + \sum_{i \in B} M_i \Delta x_i
< \sum_{i \in A} \frac{\epsilon}{2} \Delta x_i + \sum_{i \in B} M \cdot \frac{\epsilon}{4M|E|}
= \frac{\epsilon}{2} + |B| \cdot \frac{\epsilon}{4|E|}
\leq \epsilon
\]

\( \square \)