1. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be differentiable on $(0, \infty)$. Assume that $f$ is bounded and that $\lim_{x \to \infty} f'(x)$ exists. Prove that $\lim_{x \to \infty} f'(x) = 0$. (Tip: the existence of the limit is an important assumption).

**Solution:** Towards a contradiction, suppose that $\lim_{x \to \infty} f'(x) \neq 0$. Since $\lim_{x \to \infty} f'(x)$ exists, it follows from the definition of limits at infinity that there is $p > 0$ and $r \in \mathbb{R}$ such that $|f'(x)| > p$ whenever $x > r$. Set $M = \sup \{|f(x)| : x > 0\}$. Pick $h > \frac{2M}{p}$. By the Mean Value Theorem, there is $t \in (r, r + h)$ with

$$|f'(t)| = \left| \frac{f(r + h) - f(r)}{h} \right| \leq \frac{2M}{h} < p,$$

a contradiction.
2. Assume we know that the derivative of $\sin(x)$ is $\cos(x)$ and that the derivative of $\cos(x)$ is $-\sin(x)$. Prove that

$$1 - \frac{x^2}{2} \leq \cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

for all $x \in [0, \frac{\pi}{2}]$.

**Solution:** Set $f(x) = \cos(x)$. Then

\[
\begin{align*}
    f(0) &= \cos(0) = 1 \\
    f'(0) &= -\sin(0) = 0 \\
    f''(0) &= -\cos(0) = -1 \\
    f'''(0) &= \sin(0) = 0,
\end{align*}
\]

and $f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 = 1 - \frac{x^2}{2}$. Fix a value of $x$ in $[0, \frac{\pi}{2}]$. Then Taylor’s Theorem tells us that there is $t \in [0, \frac{\pi}{2}]$ with

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{f^{(4)}(t)}{4!}x^4.$$

For $t \in [0, \frac{\pi}{2}]$ we have $0 \leq f^{(4)}(t) = \sin(t) \leq 1$. Applying this inequality to the above equation yields

$$1 - \frac{x^2}{2} \leq \cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

$\square$
3. Let $f, \alpha : [-1, 1] \to \mathbb{R}$ be functions such that $f$ is bounded and $\alpha$ is monotone increasing. Assume that $f$ is continuous on $[-1, 0) \cup (0, 1]$ and that both $f$ and $\alpha$ have a simple discontinuity at 0. Additionally assume that $f(0^+) = f(0)$ and $\alpha(0^-) = \alpha(0)$. Prove that $f$ is integrable on $[-1, 1]$ with respect to $\alpha$.

**Solution:** Let $\epsilon > 0$. Set $M = \sup \{|f(x)| : x \in [-1, 1]\}$. Since $\alpha(0^-) = \alpha(0)$, there is $u < 0$ with $\alpha(0^-) - \alpha(u) < \epsilon$. Since $f$ is continuous on $[-1, 0) \cup (0, 1]$ and $f(0^+) = f(0)$, we have that the restriction of $f$ to $[-1, u] \cup [0, 1]$ is continuous. Since $[-1, u] \cup [0, 1]$ is compact, the restriction of $f$ to $[-1, u] \cup [0, 1]$ is uniformly continuous. So there is $\delta > 0$ so that for all $x, t \in [-1, u] \cup [0, 1]$ with $|x - t| < \delta$ we have $|f(x) - f(t)| < \epsilon$.

Now let $P$ be a partition of $[-1, 1]$ such that $u, 0 \in P$, $P \cap (u, 0) = \emptyset$, and $\Delta x_i < \delta$ whenever $x_i \neq 0$. Let $1 \leq j \leq n - 1$ be such that $x_j = 0$. Then $x_{j-1} = u$ and by construction $\Delta \alpha_j = \alpha(0) - \alpha(u) < \epsilon$. For $i \neq j$, since $\Delta x_i < \delta$ we have $M_i - m_i \leq \epsilon$. Setting $I = \{1, 2, \ldots, n\} \setminus \{j\}$ we have

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha_i$$

$$= \sum_{i \in I} (M_i - m_i) \Delta \alpha_i + (M_j - m_j) \Delta \alpha_j$$

$$\leq \sum_{i \in I} \epsilon \Delta \alpha_i + 2M \cdot \epsilon$$

$$\leq \epsilon (\alpha(1) - \alpha(-1)) + 2M \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that $f$ is integrable with respect to $\alpha$. \qed