1. Define \( f : [0, \infty) \to \mathbb{R} \) by
\[
f(x) = \lim_{n \to \infty} \int_0^x \sqrt{\frac{1}{n} + t^4} \, dt.
\]
Find, with justification, a formula for \( f(x) \) that does not involve any integrals or limits.

**Solution:** Fix \( x \geq 0 \). Set \( g_n(t) = \sqrt{\frac{1}{n} + t^4} \). We claim that \( (g_n) \) converges uniformly on \([0, x]\) to \( g(t) = t^2 \). Let \( \epsilon > 0 \). Since \( \sqrt{t} \) is a continuous function, it is uniformly continuous on \([0, x^4 + 1]\). So there is \( \delta > 0 \) such that
\[
\forall s, t \in [0, x^4 + 1] \quad |s - t| < \delta \implies |\sqrt{s} - \sqrt{t}| < \epsilon.
\]
Now pick an integer \( N > \frac{1}{\delta} \). Then for any \( n \geq N \) and \( t \in [0, x] \) we have that \( t^4, \frac{1}{n} + t^4 \in [0, x^4 + 1] \) and
\[
\left| t^4 - \left( \frac{1}{n} + t^4 \right) \right| = \frac{1}{n} \leq \frac{1}{N} < \delta,
\]
and hence
\[
|g(t) - g_n(t)| = \left| t^2 - \sqrt{\frac{1}{n} + t^4} \right| = \left| \sqrt{t^4} - \sqrt{\frac{1}{n} + t^4} \right| < \epsilon.
\]
We conclude \( (g_n) \) converges uniformly on \([0, x]\) to \( g \). Finally, Theorem 7.16 and the Fundamental Theorem of Calculus (Theorem 6.21) imply
\[
f(x) = \lim_{n \to \infty} \int_0^x g_n(t) \, dt = \int_0^x g(t) \, dt = \int_0^x t^2 \, dt = \frac{1}{3} x^3.
\]
2. Let \((a_n)\) and \((b_n)\) be sequences of positive real numbers. Assume that \(0 \leq a_n \leq 1/2\) and \(\lim_{n \to \infty} a_n = 0\). Let \(f_n : [0,1] \to \mathbb{R}\) be linear on the intervals \([0, a_n], [a_n, 2a_n], [2a_n, 1]\) and satisfy \(f_n(0) = f_n(2a_n) = f_n(1) = 0\) and \(f_n(a_n) = b_n\).

(a) Find the pointwise limit \(f : [0,1] \to \mathbb{R}\) of the sequence \((f_n)\).

(b) Find necessary and sufficient conditions on \((b_n)\) so that \((f_n)\) converges uniformly on \([0,1]\).

**Solution:**
(a) Since \(f_n(0) = 0\) for every \(n\), we have \(f(0) = 0\). Now consider \(x \in (0,1]\). Since \(a_n \to 0\), there is \(N\) with \(\forall n \geq N\) \(a_n < \frac{x}{2}\). So for \(n \geq N\), we have \(2a_n \leq x \leq 1\) and hence \(f_n(x) = 0\). Thus the pointwise limit \(f : [0,1] \to \mathbb{R}\) is given by \(f(x) = 0\) for all \(x \in [0,1]\).

(b) We claim that \((f_n)\) converges uniformly on \([0,1]\) if and only if \(\lim_{n \to \infty} b_n = 0\). First assume \(\lim b_n = 0\). Fix \(\epsilon > 0\). Pick \(N\) with \(\forall n \geq N\) \(b_n < \epsilon\). Since \(f_n(x) \in [0,b_n]\) for all \(x \in [0,1]\), we have that for all \(n \geq N\) and \(x \in [0,1]\)

\[|f_n(x) - f(x)| = |f_n(x)| \leq b_n < \epsilon.\]

Thus \((f_n)\) converges to \(f\) uniformly on \([0,1]\).

Now suppose that \((f_n)\) converges uniformly to \(f\) on \([0,1]\). We will show \(\lim b_n = 0\). Let \(\epsilon > 0\) and pick \(N\) with the property that for all \(n \geq N\) and all \(x \in [0,1]\) we have \(|f_n(x) - f(x)| < \epsilon\).

In particular, by plugging in \(x = a_n\) we obtain \(|b_n - 0| = |f_n(a_n) - f(a_n)| < \epsilon\) for all \(n \geq N\).

We conclude that \(\lim b_n = 0\). \(\Box\)
3. Let $F$ be an equicontinuous collection of functions mapping $[a, b]$ into $[-M, M]$. Prove that if $\phi : \mathbb{R} \to \mathbb{R}$ is continuous then the collection $H = \{ \phi \circ f : f \in F \}$ is equicontinuous.

**Solution:** Let $\epsilon > 0$. Since $\phi$ is continuous and $[-M, M]$ is compact, $\phi$ is uniformly continuous on $[-M, M]$. So there is $\kappa > 0$ with

$$\forall s, t \in [-M, M] \ |s - t| < \kappa \implies |\phi(s) - \phi(t)| < \epsilon.$$ 

Since $F$ is uniformly continuous, there is $\delta > 0$ so that

$$\forall x, y \in [a, b] \ \forall f \in F \ |x - y| < \delta \implies |f(x) - f(y)| < \kappa.$$ 

So if $x, y \in [a, b]$ satisfy $|x - y| < \delta$ and $f \in F$, then we have $s = f(x)$ and $t = f(y)$ satisfy $|s - t| < \kappa$ and thus

$$|\phi \circ f(x) - \phi \circ f(y)| = |\phi(s) - \phi(t)| < \epsilon.$$ 

We conclude that $H$ is equicontinuous. \qed