UNIFORM CONVERGENCE AND INTEGRATION: A COUNTER-EXAMPLE

This note is motivated by the question: if α_n converges uniformly to α on $[a, b]$ is it true that $\int_a^b f\ d\alpha = \lim_{n\to\infty} \int_a^b f\ d\alpha_n$? We construct an example showing the answer is negative, in contrast to Theorem 7.16 in Rudin's book. Specifically, we prove the following.

Lemma 0.1. *There exists a sequence of monotone increasing functions $\alpha_n : [0, 1] \to \mathbb{R}$ that converge uniformly on $[0, 1]$ to a monotone increasing function $\alpha : [0, 1] \to \mathbb{R}$ and there exists a function $f : [0, 1] \to \mathbb{R}$ such that f is integrable on $[0, 1]$ with respect to α_n for every n and $\int_0^1 f\ d\alpha_n = 0$ but f is not integrable on $[0, 1]$ with respect to α.*

The construction of these functions is a bit long and technical. The basic idea is that f will be 0 everywhere outside the Cantor set E and will be discontinuous at every point of E, and for every n the function α_n will be such that E can be covered with a finite number of intervals where the total cumulative change of α_n on these intervals can be made arbitrarily small, while the same will not be true of α. Each function α_n will be piecewise linear, and we will adjust α_n to obtain α_{n+1} by increasing the rate of change of α_n on a collection of intervals covering E (as n grows these intervals will have smaller width but be more numerous). The remainder of this note is devoted to providing a careful proof of the above fact.

Set $T_0 = \{0\}$, inductively define $T_{n+1} = \{3k : k \in T_n\} \cup \{3k + 2 : k \in T_n\}$, and set $T = \bigcup_{n\in\mathbb{N}} T_n$.

Lemma 0.2. $|T_n| = 2^n$ *and* $T \cap \{0, 1, \ldots , 3^n\} = T_n$.

Proof. Since T_n consists of integers, the sets $\{3k : k \in T_n\}$ and $\{3k + 2 : k \in T_n\}$ are disjoint and each has cardinality $|T_n|$. So $|T_{n+1}| = 2|T_n|$. Additionally, $|T_0| = 1 = 2^0$. So it follows from induction that $|T_n| = 2^n$ for every n.

It suffices to show by induction that $T \cap \{0, 1, \ldots , 3^n\} \subseteq T_n \subseteq \{0, 1, \ldots , 3^n - 1\}$. The base case $n = 0$ is clear. So assume this is true for n. Then

$$T_{n+1} \subseteq \{0, 1, \ldots , 3 \cdot (3^n - 1) + 2\} \subseteq \{0, 1, \ldots , 3^{n+1} - 1\}.$$

Additionally, if $t \in T \cap \{0, 1, \ldots , 3^{n+1}\}$ then either $t = 0 \in T_0 \subseteq T_{n+1}$ or else by definition of T there is $t' \in T$ with $t \in \{3t', 3t' + 2\}$. It follows that $t' \in T \cap \{0, 1, \ldots , 3^n\}$, so by our inductive assumption $t' \in T_n$ and therefore $t \in T_{n+1}$. This completes the inductive step. □

Let E denote the Cantor set. Recall that $E = \bigcap_{n=0}^{\infty} E_n$ where the E_n's are defined inductively as follows: $E_0 = [0, 1]$, in general E_n is a finite union of pairwise disjoint intervals, and E_{n+1} is obtained from E_n by removing the middle-third of each those intervals.

Lemma 0.3. *For every* $n \geq 0$

$$E_n = \bigcup_{k\in T_n} \left[\frac{k}{3^n}, \frac{k+1}{3^n}\right] \quad \textit{and} \quad E_n \setminus E_{n+1} = \bigcup_{k\in T_n} \left(\frac{k+1/3}{3^n}, \frac{k+2/3}{3^n}\right).$$

Proof. When $n = 0$ we have $T_0 = \{0\}$ so $\bigcup_{k\in T_n}[\frac{k}{3^n}, \frac{k+1}{3^n}] = [0,1] = E_0$. Now inductively assume that $E_n = \bigcup_{k\in T_n}[\frac{k}{3^n}, \frac{k+1}{3^n}]$. Since E_{n+1} is obtained from E_n by removing the middle-third of each of the above intervals, we see $E_n \setminus E_{n+1} = \bigcup_{k\in T_n}(\frac{k+1/3}{3^n}, \frac{k+2/3}{3^n})$ and

$$\begin{aligned}
E_{n+1} &= \bigcup_{k\in T_n}\left[\frac{k}{3^n}, \frac{k+1/3}{3^n}\right] \cup \left[\frac{k+2/3}{3^n}, \frac{k+1}{3^n}\right] \\
&= \bigcup_{k\in T_n}\left[\frac{3k}{3^{n+1}}, \frac{3k+1}{3^{n+1}}\right] \cup \left[\frac{3k+2}{3^{n+1}}, \frac{3k+3}{3^{n+1}}\right] \\
&= \bigcup_{k'\in\{3k:k\in T_n\}\cup\{3k+2:k\in T_n\}}\left[\frac{k'}{3^{n+1}}, \frac{k'+1}{3^{n+1}}\right] \\
&= \bigcup_{k'\in T_{n+1}}\left[\frac{k'}{3^{n+1}}, \frac{k'+1}{3^{n+1}}\right].
\end{aligned}$$

□

Fix an $\epsilon \in (0,1)$. For $t \in T$ and $x \in [0,1]$ define

$$\beta_\epsilon(t+x) = \begin{cases} \frac{1-\epsilon}{2+\epsilon}\cdot x & \text{if } x \in [0,1/3] \\ 2\cdot\frac{1-\epsilon}{2+\epsilon}(0.5-x) & \text{if } x \in (1/3,2/3) \\ \frac{1-\epsilon}{2+\epsilon}\cdot(x-1) & \text{if } x \in [2/3,1] \end{cases}$$

and set $\beta_\epsilon(y) = 0$ for all $y \in [0,+\infty) \setminus \{t+x : t \in T,\ x \in [0,1]\}$.

Lemma 0.4. *Let $m \geq 0$ and $k \in T_m$. Then the function $\beta_\epsilon(3^n x)$ has constant value 0 for $x \in (\frac{k+1/3}{3^m}, \frac{k+2/3}{3^m})$ when $n > m$ and has 0 net change on the interval $[\frac{k}{3^m}, \frac{k+1}{3^m}]$ when $n \geq m$.*

Proof. If $x \in (\frac{k+1/3}{3^m}, \frac{k+2/3}{3^m})$ then by Lemma 0.3 $x \in E_m \setminus E_{m+1} \subseteq [0,1] \setminus E_n$. On the other hand, if $\beta_\epsilon(3^n x) \neq 0$ then there is $k' \in T$ with $3^n x \in [k', k'+1]$, hence $x \in [\frac{k'}{3^n}, \frac{k'+1}{3^n}]$ and $k' \leq 3^n x \leq 3^n$. Therefore $k' \in T_n$ by Lemma 0.2 and $x \in E_n$ by Lemma 0.3. This proves the first claim. Finally, the last claim holds since $3^n \cdot \frac{k}{3^m} = 3^{n-m}k \in T$ and $3^n \cdot \frac{k+1}{3^m}$ is 1 plus

$$\begin{aligned}
3^{n-m}k + 3^{n-m} - 1 &= 3^{n-m}k + 3^{n-m-1}\cdot 2 + 3^{n-m-2}\cdot 2 + \ldots + 3\cdot 2 + 2 \\
&= 3(3(\cdots(3(3k+2)+2)\cdots+2)+2)+2 \in T.
\end{aligned}$$

□

Lemma 0.5. *Let $\phi : [0,1] \to \mathbb{R}$ be a linear increasing function with net change $\lambda = \phi(1) - \phi(0)$. Then the function $\phi(x) + \lambda\cdot\beta_\epsilon(x)$ also increases by λ on $[0,1]$ and is linear on each of the intervals $[0,1/3]$, $[1/3,2/3]$, $[2/3,1]$. Moreover, it increases by $\lambda(2+\epsilon)^{-1}$ on each of the intervals $[0,1/3]$ and $[2/3,1]$ and increases by $\lambda\epsilon(2+\epsilon)^{-1}$ on the interval $[1/3,2/3]$.*

Proof. Since ϕ is linear we deduce from the point-slope formula that

$$\phi(x) = \phi(0) + \lambda x = \phi(1) + \lambda(x-1) = \phi(0.5) + \lambda(x-0.5)$$

for all $x \in [0,1]$. So for $x \in [0,1/3]$ we have

$$\phi(x) + \lambda\cdot\beta_\epsilon(x) = \phi(0) + \lambda x + \lambda\cdot\frac{1-\epsilon}{2+\epsilon}\cdot x = \phi(0) + \lambda\cdot\frac{3}{2+\epsilon}\cdot x,$$

implying an increase by $\lambda(2+\epsilon)^{-1}$ on $[0,1/3]$. Similarly, for $x \in [2/3,1]$ we have

$$\phi(x)+\lambda\cdot\beta_\epsilon(x)=\phi(1)+\lambda(x-1)+\lambda\cdot\frac{1-\epsilon}{2+\epsilon}\cdot(x-1)=\phi(1)+\lambda\cdot\frac{3}{2+\epsilon}\cdot(x-1),$$

implying an increase by $\lambda(2+\epsilon)^{-1}$ on $[2/3,1]$. Lastly, for $x\in[1/3,2/3]$

$$\phi(x)+\lambda\cdot\beta_\epsilon(x)=\phi(0.5)+\lambda(x-0.5)+2\lambda\cdot\frac{1-\epsilon}{2+\epsilon}\cdot(0.5-x)=\phi(0.5)+\lambda\cdot\frac{3\epsilon}{2+\epsilon}\cdot(x-0.5),$$

implying an increase by $\lambda\epsilon(2+\epsilon)^{-1}$ on $[1/3,2/3]$. □

For $n\geq 1$ choose $\epsilon_n>0$ small enough that $2(2+\epsilon_n)^{-1}>2^{-2^{-n}}$. For $x\in[0,1]$ define

$$\alpha(x)=x+\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}(2+\epsilon_i)^{-1}\right)\cdot\beta_{\epsilon_k}(3^{k-1}x)$$

and let α_n be the n^{th} partial sum

$$\alpha_n(x)=x+\sum_{k=1}^{n}\left(\prod_{i=1}^{k-1}(2+\epsilon_i)^{-1}\right)\cdot\beta_{\epsilon_k}(3^{k-1}x).$$

Lemma 0.6. *Let $n\geq 0$. The interval $[0,1]$ is the union of the pairwise disjoint intervals*

$$\left[\frac{k}{3^n},\frac{k+1}{3^n}\right]\quad(k\in T_n)\ \textit{and}\ \left(\frac{k+1/3}{3^m},\frac{k+2/3}{3^m}\right)\quad(0\leq m<n,\ k\in T_m).$$

The function α_n is monotone increasing and is linear on each of the above intervals. Moreover, α_n increases by $\prod_{i=1}^{n}(2+\epsilon_i)^{-1}$ on each of the intervals $[\frac{k}{3^n},\frac{k+1}{3^n}]$, $k\in T_n$, and increases by $\epsilon_m\cdot\prod_{i=1}^{m}(2+\epsilon_i)^{-1}$ on each of the intervals $(\frac{k+1/3}{3^m},\frac{k+2/3}{3^m})$, $0\leq m<n$, $k\in T_m$.

Proof. We have

$$(E_0\setminus E_1)\cup(E_1\setminus E_2)\cup\cdots\cup(E_{n-1}\setminus E_n)\cup E_n=E_0=[0,1],$$

with the sets appearing in the above union disjoint with one another. By Lemma 0.3 E_n is the union of the disjoint intervals $[\frac{k}{3^n},\frac{k+1}{3^n}]$ for $k\in T_n$, and for each $m<n$ the set $E_m\setminus E_{m+1}$ is the union of the disjoint intervals $(\frac{k+1/3}{3^m},\frac{k+2/3}{3^m})$ for $k\in T_m$. This proves the first claim.

We prove the remaining claims by induction. First consider the base case $n=1$. The function $\phi(x)=x$ increases by $\lambda=1$ on $[0,1]$, so by Lemma 0.5 $\alpha_1(x)=x+\beta_{\epsilon_1}(x)$ is linear on each interval $[0,1/3]$, $[1/3,2/3]$, $[2/3,1]$, increases by $(2+\epsilon_1)^{-1}$ on each of the intervals $[0,1/3]$ and $[2/3,1]$, and increases by $\epsilon_1(2+\epsilon_1)^{-1}$ on $(1/3,2/3)$.

Now inductively assume that α_n increases by $\prod_{i=1}^{n}(2+\epsilon_i)^{-1}$ on each of the intervals $[\frac{k}{3^n},\frac{k+1}{3^n}]$, $k\in T_n$, increases by $\epsilon_m\cdot\prod_{i=1}^{m}(2+\epsilon_i)^{-1}$ on each of the intervals $(\frac{k+1/3}{3^m},\frac{k+2/3}{3^m})$, $0\leq m<n$, $k\in T_m$, and is moreover linear on each of these intervals.

Consider an interval $[\frac{k}{3^n}, \frac{k+1}{3^n}]$ where $k \in T_n$. For $u \in [0,1]$ define $\phi(u) = \alpha_n(\frac{k+u}{3^n})$. Then ϕ is linear and increases by $\prod_{i=1}^{n}(2+\epsilon_i)^{-1}$ on $[0,1]$. Notice that

$$\begin{aligned}\alpha_{n+1}\left(\frac{k+u}{3^n}\right) &= \alpha_n\left(\frac{k+u}{3^n}\right) + \left(\prod_{i=1}^{n}(2+\epsilon_i)^{-1}\right)\beta_{\epsilon_{n+1}}\left(3^n \cdot \frac{k+u}{3^n}\right) \\ &= \phi(u) + \left(\prod_{i=1}^{n}(2+\epsilon_i)^{-1}\right)\beta_{\epsilon_{n+1}}(k+u) \\ &= \phi(u) + \left(\prod_{i=1}^{n}(2+\epsilon_i)^{-1}\right)\beta_{\epsilon_{n+1}}(u),\end{aligned}$$

where the final equality holds since $k \in T$. By combining the final line of the above equation with Lemma 0.5, we find that α_{n+1} increases by $\prod_{i=1}^{n+1}(2+\epsilon_i)^{-1}$ on each of the intervals (A) $[\frac{k}{3^n}, \frac{k+1/3}{3^n}] = [\frac{3k}{3^{n+1}}, \frac{3k+1}{3^{n+1}}]$ and (B) $[\frac{k+2/3}{3^n}, \frac{k+1}{3^n}] = [\frac{3k+2}{3^{n+1}}, \frac{3k+3}{3^{n+1}}]$, increases by $\epsilon_{n+1}\prod_{i=1}^{n+1}(2+\epsilon_i)^{-1}$ on (C) $[\frac{k+1/3}{3^n}, \frac{k+2/3}{3^n}]$, and is linear on each of these intervals. Notice that the intervals (A) and (B) just described, as $k \in T_n$ varies, coincide with the intervals $[\frac{k'}{3^{n+1}}, \frac{k'+1}{3^{n+1}}]$ as $k' \in T_{n+1}$ varies (by Lemma 0.3), and the intervals (C), as $k \in T_n$ varies, coincide with the intervals $(\frac{k+1/3}{3^m}, \frac{k+2/3}{3^m})$ as $k \in T_m$ varies with $m = n < n+1$.

Next consider an interval $(\frac{k+1/3}{3^m}, \frac{k+2/3}{3^m})$ where $k \in T_m$ and $m < n$. By our inductive hypothesis α_n is linear on this interval and increases by $\epsilon_m \cdot \prod_{i=1}^{m}(2+\epsilon_i)^{-1}$ on this interval. Since Lemma 0.4 tells us $\beta_{\epsilon_{n+1}}(3^n x) = 0$ for all x in this interval, it follows that α_{n+1} is equal to α_n on this interval. Therefore α_{n+1} is linear on $(\frac{k+1/3}{3^m}, \frac{k+2/3}{3^m})$ and increases by $\epsilon_m \cdot \prod_{i=1}^{m}(2+\epsilon_i)^{-1}$ on this interval.

By induction, we conclude that the claim holds for all n. □

Lemma 0.7. *The functions (α_n) converge uniformly to α on $[0,+\infty)$.*

Proof. Notice that for every $x \geq 0$

$$|\beta_\epsilon(x)| \leq \frac{1-\epsilon}{2+\epsilon} \cdot \frac{1}{3} < \frac{1}{2} \cdot \frac{1}{3}.$$

So uniform convergence follows from the Weierstrass M-test (Theorem 7.10) since $\sum_{k=1}^{\infty} \frac{1}{3} \cdot \frac{1}{2^k}$ converges and

$$\left|\left(\prod_{i=1}^{k-1}(2+\epsilon_i)^{-1}\right) \cdot \beta_{\epsilon_k}(3^{k-1}x)\right| \leq 2^{-(k-1)}|\beta_{\epsilon_k}(3^{k-1}x)| \leq \frac{1}{3} \cdot \frac{1}{2^k}.$$ □

Corollary 0.8. *α is monotone increasing and for every $m \geq 0$ and $k \in T_m$ we have*

$$\alpha\left(\frac{k+1}{3^m}\right) - \alpha\left(\frac{k}{3^m}\right) = \prod_{i=1}^{m}(2+\epsilon_i)^{-1}.$$

Proof. Lemma 0.6 tells us that each α_n is monotone increasing. So if $y > x$ then $\alpha_n(y) - \alpha_n(x) \geq 0$ for every n and $\alpha(y) - \alpha(x) = \lim_{n\to\infty}(\alpha_n(y) - \alpha_n(x)) \geq 0$. Thus α is monotone increasing. Next, from Lemma 0.6 we obtain $\alpha_m(\frac{k+1}{3^m}) - \alpha_m(\frac{k}{3^m}) = \prod_{i=1}^{m}(2+\epsilon_i)^{-1}$, and Lemma 0.4 implies that $\beta_{\epsilon_n}(3^{n-1} \cdot \frac{k+1}{3^m}) - \beta_{\epsilon_n}(3^{n-1} \cdot \frac{k}{3^m}) = 0$

for all $n > m$. Consequently, for all $n \geq m$

$$\alpha_n\left(\frac{k+1}{3^m}\right) - \alpha_n\left(\frac{k}{3^m}\right) = \alpha_m\left(\frac{k+1}{3^m}\right) - \alpha_m\left(\frac{k}{3^m}\right) = \prod_{i=1}^{m}(2+\epsilon_i)^{-1}.$$

Taking the limit as $n \to \infty$ completes the proof. □

Define $f : [0,1] \to \mathbb{R}$ by the rule

$$f(x) = \begin{cases} 1 & \text{if } \exists n\ \exists k \in T_n\ x = \frac{k}{3^n} \\ 1 & \text{if } \exists n\ \exists k \in T_n\ x = \frac{k+1}{3^n} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\frac{k}{3^n}, \frac{k+1}{3^n}$ are in the Cantor set E for every n and every $k \in T_n$. Thus $f(x) = 0$ for all $x \in [0,1] \setminus E$.

Lemma 0.9. *For every $q \geq 0$, f is integrable on $[0,1]$ with respect to α_q and $\int_0^1 f\ d\alpha_q = 0$.*

Proof. By Lemma 0.6 we can break $[0,1]$ into a finite number of sub-intervals on which α_q is linear. Letting A be the maximum slope among these linear pieces, we have $|\alpha_q(x) - \alpha_q(y)| \leq A|x-y|$ for all $x, y \in [0,1]$. Let $\epsilon > 0$ and pick an integer m with $(\frac{2}{3})^m < \frac{\epsilon}{A}$. Consider the partition $P = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$ where $n = 3^m$ and $x_i = \frac{i}{3^m}$ for all $0 \leq i \leq n$. Set $I = \{k + j : k \in T_m,\ j = -1, 0, 1\}$ and notice that for $1 \leq i \leq n$ Lemma 0.3 implies that $[x_{i-1}, x_i] \cap E_m \neq \varnothing \Leftrightarrow i \in I$. Since f has constant value 0 on $[0,1] \setminus E_m \subseteq [0,1] \setminus E$, it follows that $M_i = 0$ for all $i \in \{1, 2, \ldots, n\} \setminus I$. Additionally, for every $1 \leq i \leq n$ we have $0 \leq m_i \leq M_i \leq 1$. So

$$0 \leq L(P, f, \alpha_q) \leq U(P, f, \alpha_q) \leq \sum_{i \in I} \Delta(\alpha_q)_i \leq \sum_{i \in I} A \cdot \Delta x_i.$$

For every i we have $\Delta x_i = \frac{1}{3^m}$ and Lemma 0.2 gives $|I| \leq 3|T_m| = 3 \cdot 2^m$. Therefore

$$U(P, f, \alpha_q) \leq |I| \cdot A \cdot \frac{1}{3^m} = 3A \cdot \left(\frac{2}{3}\right)^m < \epsilon.$$

We conclude that f is integrable on $[0,1]$ with respect to α_q and $\int_0^1 f\ d\alpha_q = 0$. □

Lemma 0.10. *f is not integrable on $[0,1]$ with respect to α.*

Proof. Let P be any partition of $[0,1]$, say $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$. Let i be the set of indices $1 \leq i \leq n$ satisfying $[x_{i-1}, x_i] \cap E \neq \varnothing$, and let K be the compact set given by the union of the intervals $[x_{i-1}, x_i]$ for $i \in \{1, 2, \ldots, n\} \setminus I$. Then $\bigcap_{m \in \mathbb{N}}(E_m \cap K) = E \cap K = \varnothing$, so the Corollary to Theorem 2.36 implies that there is m with $E_m \cap K = \varnothing$. In other words, $E_m \subseteq \bigcup_{i \in I}[x_{i-1}, x_i]$. By choosing a larger m if necessary, we can assume that $\frac{1}{3^m} < \min\{\Delta x_i : 1 \leq i \leq n\}$.

If $i \in I$ then $E_m \cap [x_{i-1}, x_i] \supseteq E \cap [x_{i-1}, x_i] \neq \varnothing$, so by Lemma 0.3 there is $k \in T_m$ with $[\frac{k}{3^m}, \frac{k+1}{3^m}] \cap [x_{i-1}, x_i] \neq \varnothing$. Since $\frac{1}{3^m} < \Delta x_i$, we have either $\frac{k}{3^m}$ or $\frac{k+1}{3^m}$ lies in $[x_{i-1}, x_i]$. So f attains a value of 1 somewhere on $[x_{i-1}, x_i]$ and thus $M_i = 1$ for every $i \in I$. Therefore $U(P, f, \alpha) \geq \sum_{i \in I} \Delta(\alpha)_i$. On the other hand, for every $1 \leq i \leq n$ the set $[x_{i-1}, x_i]$ is uncountable while f has value 1 at only a countable number of points. So f attains a value of 0 on $[x_{i-1}, x_i]$ and $m_i = 0$ for every $1 \leq i \leq n$. Therefore $L(P, f, \alpha) = 0$.

From Lemma 0.3 we have

$$\bigcup_{i\in I}[x_{i-1},x_i] \supseteq E_m = \bigcup_{k\in T_m}\left[\frac{k}{3^m},\frac{k+1}{3^m}\right].$$

Therefore, since α is monotone increasing,

$$\sum_{i\in I}\Delta(\alpha)_i \geq \sum_{k\in T_m}\alpha\left(\frac{k+1}{3^m}\right)-\alpha\left(\frac{k}{3^m}\right),$$

and Corollary 0.8 and Lemma 0.2 imply

$$\sum_{k\in T_m}\alpha\left(\frac{k+1}{3^m}\right)-\alpha\left(\frac{k}{3^m}\right) = |T_m|\cdot\prod_{i=1}^{m}(2+\epsilon_i)^{-1} = \prod_{i=1}^{m}2\cdot(2+\epsilon_i)^{-1}.$$

Recalling that we chose ϵ_i to satisfy $2(2+\epsilon_i)^{-1} > 2^{-2^{-i}}$, we conclude that

$$\sum_{i\in I}\Delta(\alpha)_i \geq \prod_{i=1}^{m}2^{-2^{-i}} = 2^{-\sum_{i=1}^{m}2^{-i}} > 2^{-1} = \frac{1}{2}.$$

Therefore $U(P,f,\alpha) \geq \sum_{i\in I}\Delta(\alpha)_i > \frac{1}{2}$ while $L(P,f,\alpha)=0$. We conclude that f is not integrable on $[0,1]$ with respect to α. □