140C Office Hours
Why is $\|A\|^{-1}$ used in Thm 9.8?

$\|A\|$ only bounds how much $A$ can lengthen vectors. $\frac{1}{\|A\|^{-1}}$ bounds how much $A$ shrinks vectors (and the concern for invertibility is that a nonzero vector gets shrunk to the 0 vector).

Suppose $A \in L(R^n)$ is invertible for all $y \in R^n$:

$$|A^{-1} y| \leq \|A^{-1}\| \|y\|$$

Setting $x = A^{-1} y$ we obtain:

$$|x| \leq \|A^{-1}\| |Ax|$$

Meaning:

$$|Ax| \geq \frac{1}{\|A^{-1}\|} |x|$$

This holds for all $x \in R^n$.

In general, (still assuming $A$ invertible):

$$\forall x \in R^n \quad \frac{1}{\|A\|} |x| \leq |Ax| \leq \|A\| |x|$$

and $\|A\|$ and $\|A^{-1}\|$ are the smallest real numbers for which the above statement is true.

In Thm 9.8 require $\|B - A\| < \frac{1}{\|A^{-1}\|}$.
Why are the two formulas for \( \|A\| \) equal?

By definition, \( \|A\| = \sup_{x \in \mathbb{R}^n, \|x\| = 1} |Ax| \).

Clearly, \( \|A\| \geq \sup_{x \in \mathbb{R}^n, \|x\| = 1} |Ax| \) since \( \{x \in \mathbb{R}^n : \|x\| = 1\} \subseteq \{x \in \mathbb{R}^n : \|x\| \leq 1\} \).

On the other hand, consider any \( x \in \mathbb{R}^n \) with \( \|x\| \leq 1 \).

Case 1: \( x = 0 \). Then \( Ax = 0 \) so \( |Ax| = 0 \leq \sup_{y \in \mathbb{R}^n, \|y\| = 1} |Ay| \).

Case 2: \( x \neq 0 \). Set \( t = \frac{1}{\|x\|} \). Then \( \|tx\| = 1 \) and since \( t \geq 1 \), we have

\[
|Ax| \leq t |Ax| = |Ax| = \sup_{y \in \mathbb{R}^n, \|y\| = 1} |Ay|.
\]

So \( \sup_{x \in \mathbb{R}^n, \|x\| = 1} |Ay| \) is an upper bound to \( \sup_{x \in \mathbb{R}^n, \|x\| = 1} |Ax| \).

Therefore, \( \|A\| \leq \sup_{y \in \mathbb{R}^n, \|y\| = 1} |Ay| \). We conclude

\[
\|A\| = \sup_{y \in \mathbb{R}^n, \|y\| = 1} |Ay|.
\]
Why is $\text{GL}(\mathbb{R}^n)$ open?

Recall if $(X, d)$ metric space and $U \subseteq X$, then

$U$ is open if $\forall x \in U \exists r > 0 \quad B_r(x) \subseteq U$.

$\text{GL}(\mathbb{R}^n)$ is open by Theorem 9.8(�) since

for every $A \in \text{GL}(\mathbb{R}^n)$

$$B_{\frac{1}{\|A^{-1}\|}}(A) \subseteq \text{GL}(\mathbb{R}^n)$$

$\forall A \in \text{GL}(\mathbb{R}^n)$

$$B \in B_{\frac{1}{\|A^{-1}\|}}(A) \Rightarrow \|B - A\| < \frac{1}{\|A^{-1}\|} \quad \text{[Thm 9.8(�)]} \Rightarrow B \in \text{GL}(\mathbb{R}^n)$
$f : E \to \mathbb{R}, \quad E \subseteq \mathbb{R}^n$

$f$ differentiable, local max at $x$. Show $f'(x) = 0$.

Write $f(x+h) - f(x) = f'(x)h + r(h)$ where $\frac{|r(h)|}{|h|} \to 0$ as $h \to 0$.

Towards a contradiction, suppose $f'(x) \neq 0$.

Then there is $h \in \mathbb{R}^n$ with $f'(x)h \neq 0$.

By replacing $h$ with $-h$ if necessary, can assume $f'(x)h > 0$.

Set $\epsilon = \frac{1}{2|f'(x)|} f'(x)h$.

Then for all $t > 0$ close enough to $0$ so that $\frac{|r(th)|}{|th|} < \epsilon$ we have

$$f(x+th) - f(x) = tf'(x)h + r(th)$$

$$\geq tf'(x)h - |r(th)|$$

$$\geq tf'(x)h - \epsilon t|h|$$

$$= t(f'(x)h - \epsilon |h|)$$

$$\geq \frac{1}{2} tf'(x)h > 0.$$ 

Thus $x$ is not a local max, contradiction.
If \( g(x) = A f(x) \) why is \( g'(x) = A f'(x) \)?

**Claim:** If \( A \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^k) \), \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \)
and \( g(x) = A f(x) \), then \( g'(x) = A f'(x) \)
(assuming \( f'(x) \) exists)

**Pf:** We have

\[
\lim_{h \to 0} \frac{|g(x+h)-g(x)|}{|h|} = \lim_{h \to 0} \frac{|A(f(x+h)-f(x))-f'(x)h|}{|h|} \\
\leq \lim_{h \to 0} \|A\| \frac{|f(x+h)-f(x)-f'(x)h|}{|h|} = 0 \quad \square
\]
Why do Lebesgue-measurable non-Borel measurable sets exist?

**Fact:** Let \( m \) be Lebesgue measure on \( \mathbb{R}^n \). If \( A \in \mathcal{M}(m) \) and \( m(A) = 0 \) then \( B \in \mathcal{M}(m) \) for all \( B \subseteq A \).

**Pf:** Since \( 0 = m(A) = m^*(A) \) and \( m^* \) is monotone, so \( \forall B \subseteq A \) \( m^*(B) = 0 \). This means \( B \in \mathcal{M}_c(m) \subseteq \mathcal{M}(m) \) because \( B_n \to B \) where \( B_n = \emptyset, \mathcal{E} \) (because \( B_n \to B \) since \( \lim_{n \to \infty} m^*(B \Delta B_n) = m^*(B) = 0 \)).

**Fact:** If \( C \) is the Cantor set then \( m(C) = 0 \).

**Pf:** Given in class.

So \( C \) is an uncountable compact set and every subset of \( C \) is Lebesgue measurable.

However, the collection of Borel sets contained in \( C \) coincides with the Borel \( \sigma \)-algebra of \( C \) (for any metric space \( (X, d) \) the Borel \( \sigma \)-algebra is by definition the smallest \( \sigma \)-algebra containing all open sets) (has a countable dense subset).

**Fact:** If \( (X, d) \) is an uncountable, separable, complete metric space, then there exist subsets of \( X \) that are not Borel
Fact: \((X,d)\) uncountable, separable, complete. Then \(|B(x)| = |\mathbb{R}|\).

\[\text{Claim: } |\{u \leq x : u \text{ open}\}| = |\mathbb{R}|\]

\[\text{Pf: Let } X_0 \leq X \text{ be cntbl dense. Set } \gamma = \{B_q(x) : q \geq 0, q \in \mathbb{Q}, x \in X_0\}.
\]

- For any open \(U \subseteq X\) and any \(x \in U\), there is \(B_q(x') \in \gamma\) with \(x \in B_q(x') \subseteq U\).
- Enumerate \(\gamma\) as \(V_0, V_1, \ldots\).

This shows \(U \text{ open } \subseteq X \implies \exists i \in \mathbb{N} \ U = \bigcup_{i \in \mathbb{N}} U_i\).

Therefore \(|\{u \leq x : u \text{ open}\}| = |\mathbb{N}| \leq |\mathbb{R}|\).

Reverse inequality \(\leq \ldots \) ?

\[\text{Claim implies } |B(x)| = |\mathbb{R}|, \]

Let \(N = \text{Baire space} = \{x : N \rightarrow \mathbb{N}\}^3\)

Fact: Every Borel set \(A \subseteq X\) is the projection of a closed set \(B \subseteq X \times N\)

\[A = \{x \in X : \exists y \in N \ (x,y) \in B\}\]

By first claim, \(X \times N\) has at most \(|\mathbb{R}|\)-many closed sets. So by above fact \(|B(x)| \leq |\mathbb{R}|\). \(\Box\)
Continued from previous page

Fact: \(|\mathbb{R}^n\cap x| > |x| = |\mathbb{R}|\)

Pf: “>” by Math 109

\[N\text{ is a metric space: if } x,y : N \rightarrow N \]
\[d(x,y) = \inf \xi 2^{-n} : n \in \mathbb{N} \quad \forall k < n \quad x(k) = y(k) \leq 1\]

Fact: Every Borel set \(A \subseteq X\) is the projection of a closed set \(B \subseteq X \times N\)

\[(A = \exists x \in X : \exists y \in N : (x, y) \in B^3)\]

Pf Sketch: Set \(A = \exists A \subseteq X : \exists \text{ closed } B \subseteq X \times N \text{ with } \pi_x(B) = A^3\).
Check that \(A\) is a \(\sigma\)-algebra and contains every closed subset of \(X\). Since \(B(x)\) is the smallest \(\sigma\)-algebra containing all closed sets, \(B(x) \subseteq A\).
Measurable Functions

Fact: If \( f: X \rightarrow [-\infty, +\infty] \) is measurable iff
\[
\forall A \in B(\mathbb{R}) \quad f^{-1}(A) \in M
\]

In general, if \((X, M, \mu)\) and \((Y, N, \nu)\) are measure spaces then a function \( f: X \rightarrow Y \) is measurable if
\[
\forall A \in N \quad f^{-1}(A) \in M
\]

So for the definition of measurable we use in class, we take the codomain \( \mathbb{R} \) to be equipped with \( B(\mathbb{R}) \) by default.

\[
x \in X \quad \mapsto f(x) \in \mathbb{R} \quad \mapsto g(f(x)) \in \mathbb{R}
\]
The answer is yes. To see this, it suffices to show that whenever a sequence \((x_n)\) in \([0, 1]\) converges to \(x\), \(g(x_n) \to g(x)\) as \(n \to \infty\).

Consider such a seq. \((x_n)\). Define \(f_n(y) = f(x_n, y)\). By (b) \(f_n(y) \to f(x, y)\). Also \(\text{by } \|f_n(y)\| \leq 1\) and \(\int_0^1 dy < \infty\), so by Lebesgue Dominated Convergence Theorem

\[
g(x) = \int_0^1 f(y) dy = \int_0^1 \lim_{n \to \infty} f_n(y) dy = \lim_{n \to \infty} \int_0^1 f_n(y) dy = \lim_{n \to \infty} g(x_n). \]

\(\square\)
Fatou’s Theorem - Different Perspective

Fix \( \varepsilon > 0 \).

Define \( E_n = \{ x \in E : \inf_{m \geq n} f_m(x) > (1-\varepsilon)f(x) \} \).

Then \( E_1 \subseteq E_2 \subseteq \ldots \) and since \( f(x) = \liminf_{n \to \infty} f_n(x) \) (\( \star \)), we have \( \bigcup_{n=1}^{\infty} E_n = E. \)

Since the \( f_n \)'s are non-negative, we have

\[
\lim_{m \to \infty} \int_{E_n} f(x) \, dx \leq \int_{E_n} f_m(x) \, dx \leq \int_{E_n} f(x) \, dx + \int_{E \setminus E_n} f(x) \, dx
\]

Taking \( \liminf_{n \to \infty} \) we obtain

\[
\lim_{m \to \infty} \int_{E_n} f(x) \, dx \leq \liminf_{n \to \infty} \int_{E_n} f_m(x) \, dx
\]

Next by Theorem 11.24 and Theorem 11.3 we can take

\[
(1-\varepsilon) \int_{E} f(x) \, dx = \int_{E} (1-\varepsilon) f(x) \, dx = \lim_{n \to \infty} \int_{E} (1-\varepsilon) f(x) \, dx \leq \liminf_{n \to \infty} \int_{E} f_m(x) \, dx
\]

Now take limit as \( \varepsilon \to 0. \)

(\( \star \)) \( f(x) = \liminf_{m \to \infty} f_m(x) = \lim_{n \to \infty} \inf_{m \geq n} f_m(x) \)

Summary: If \( n \) is large enough

for “most” points \( x \in E \) we have \( (1-\varepsilon)f(x) < f_n(x) \)

and thus

\[
\int_{E} (1-\varepsilon) f(x) \, dx \leq \int_{E} f(x) \, dx
\]
Chapter 11 Problem 7

**Modified Theorem 11.33:**

Let \( a \leq b \) be real numbers, let \( \alpha : \mathbb{R} \to \mathbb{R} \) be monotone increasing. Define

\[
\alpha(x) = \begin{cases} 
\alpha(a) & \text{if } x < a \\
\alpha(x) & \text{if } a \leq x \leq b \\
\alpha(b) & \text{if } x > b
\end{cases}
\]

Let \( \mu \) be the measure obtained from using \( \alpha \), in Ex.6a.(b).

1. If \( f \in \mathcal{R}_a^b(\alpha) \) then \( f \in L([a,b], \mu) \) and \( \int_a^b f \, d\mu = \int_a^b f \, d\alpha \)

2. Suppose \( f \) is bounded. Then \( f \in \mathcal{R}_a^b(\alpha) \) iff \( f \) is left-cont. at every point \( \alpha \), is not left-cont., \( f \) is right-cont. at every point \( \alpha \), is not right-cont., and the set of points where \( \alpha \), is continuous and \( f \) is discontinuous has \( \mu \)-measure 0.