Ch. 12 - Counting Functions and Subsets

Highlights:
- Counting functions
- Counting injections
- Counting permutations
- Cardinality of the power set
- Binomial Coefficients
- Properties & formulas for binomial coefficients
- Binomial theorem

Problem types:
- Counting the number of functions with certain properties
- Counting the number of subsets with certain properties
- Proving identities involving binomial coefficients
Defn: For sets $X$ and $Y$, we let $\text{Fun}(X,Y)$ denote the set of functions from $X$ to $Y$.

Prop: If $X, Y$ are sets with $|X|=m$ and $|Y|=n$, then $|\text{Fun}(X,Y)| = n^m$.

To build a function, you must make a choice for each of the $m$-many elements of $X$; every element of $X$ has $n$-many choices for an element of $Y$.

Proof: Exercise (use induction on $m$).

Defn: For sets $X, Y$ we let $\text{Inj}(X,Y)$ denote the set of injections from $X$ to $Y$.

Prop: If $X, Y$ are sets with $|X|=m$ and $|Y|=n$, then $|\text{Inj}(X,Y)| = n(n-1)(n-2)\cdots(n-m+1)$

called a falling factorial, denoted $(n)_m$.

Intuition: Say $X = \{x_1, x_2, \ldots, x_m\}$. We can build $f \in \text{Inj}(X,Y)$ by first picking $f(x_1) \in Y$ ($n$ choices), then picking $f(x_2) \in Y - \{f(x_1)\}$ ($n-1$ choices), then picking $f(x_3) \in Y - \{f(x_1), f(x_2)\}$ ($n-2$ choices), and so on, finally picking $f(x_m) \in Y - \{f(x_1), f(x_2), \ldots, f(x_{m-1})\}$ ($n-m+1$ choices). So there are $n(n-1)(n-2)\cdots(n-m)$ choices for $f \in \text{Inj}(X,Y)$. 
Proof: Exercise

Defn: A bijection \( f: X \rightarrow X \) is called a permutation of \( X \)

Corollary: If \( |X| = n \) then the number of permutations of \( X \) is \( n! \)

Proof: Since \( X \) is finite, a previous theorem (in Ch. 11) tells us that a function \( f: X \rightarrow X \) is injective if and only if it is surjective. Thus a function \( f: X \rightarrow X \) is a bijection (permutation) if and only if it is injective. So the number of permutations is

\[
|\text{Im}(X, X)| = |X| = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = n!
\]

Prop: If \( X \) is a finite set then \( |P(X)| = 2^{\mid X \mid} \)

Intuition: To build a set \( A \in P(X) \) we must choose for each \( x \in X \) whether \( x \in A \) or \( x \notin A \). Since there are two choices per \( x \in X \), the number of choices for \( A \) is \( 2^{\mid X \mid} \). Thus \( |P(X)| = 2^{\mid X \mid} \).

Proof: Exercise

Defn: For a set \( X \) and an integer \( r \geq 0 \) we define

\[
P_r(X) = \{ A \subseteq X : |A| = r \}
\]
Example: If \( X = \{1, 2, 3\} \) then
\[
\begin{align*}
\Pr_0(X) &= \emptyset \quad (0) = 1 \\
\Pr_1(X) &= \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \quad (3) = 3 \\
\Pr_2(X) &= \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \quad (3) = 3 \\
\Pr_3(X) &= \{1, 2, 3\} \quad (3) = 1 \\
\Pr_r(X) &= \emptyset \quad \text{for } r \geq 4
\end{align*}
\]

Defn: We define the binomial coefficient \(^n\text{C}_r\) (pronounced "n choose r") to be \( |\Pr_r(X)| \) for any set \( X \) with \(|X| = n\).

Prop: For integers \( n, r \geq 0 \):
\[
\begin{align*}
1. \quad \text{if } r > n, \quad \binom{n}{r} = 0 \\
2. \quad \binom{n}{0} = \binom{n}{n} = 1 \\
3. \quad \binom{n}{r} = \binom{n}{n-r} \quad \text{when } 0 \leq r \leq n
\end{align*}
\]

Proof of (3): Let \( n, r \geq 0 \) be integers. Fix a set \( X \) with \(|X| = n\). Define \( f: \Pr_r(X) \to \Pr_{n-r}(X) \) by \( f(A) = X - A \) for \( A \in \Pr_r(X) \). Then \( f \) is a bijection (exercise) so \( |\Pr_r(X)| = |\Pr_{n-r}(X)| \). Therefore \( \binom{n}{r} = |\Pr_r(X)| = |\Pr_{n-r}(X)| = \binom{n}{n-r} \). \( \square \)

Prop: If \( 1 \leq r \leq n \) then \( \binom{n}{r} = (\binom{n-1}{r-1}) + (\binom{n-1}{r}) \)

Proof: Let \( X \) be a set with \(|X| = n \geq 1\) and pick \( x_0 \in X \). Define \( f: \Pr_r(X) \to \Pr_{r+1}(X - x_0^3) \cup \Pr_r(X - x_0^3) \) by...
\[ f(A) = \begin{cases} \sum A - 3x_0^3 & \text{if } x_0 \in A \\ A & \text{if } x_0 \notin A \end{cases} \]

for \( A \in \mathcal{P}_r(x) \) and define \( g : \mathcal{P}_{r-1}(x-x_0^3) \cup \mathcal{P}_r(x-x_0^3) \) by

\[ g(B) = \begin{cases} 0 & \text{if } B \subseteq \mathcal{P}_{r-1}(x-x_0^3) \\ B & \text{if } B \subseteq \mathcal{P}_r(x-x_0^3) \end{cases} \]

for \( B \subseteq \mathcal{P}_{r-1}(x-x_0^3) \cup \mathcal{P}_r(x-x_0^3) \). Then \( f \) and \( g \) are inverses of each other (exercise), so \( f \) is a bijection. Therefore

\[ (\mathcal{P}) = |\mathcal{P}_r(x)| = |\mathcal{P}_{r-1}(x-x_0^3) \cup \mathcal{P}_r(x-x_0^3)| \]

\[ = |\mathcal{P}_{r-1}(x-x_0^3)| + |\mathcal{P}_r(x-x_0^3)| \\ = (\mathcal{P})^{r-1} + (\mathcal{P})^{r-1}. \]

Theorem: If \( 0 \leq r \leq n \) then \( (\mathcal{P}) = \frac{n!}{r!(n-r)!} \)

Proof: Exercise (Use previous proposition and induction on \( n \))

The Binomial Theorem: Let \( a, b \in \mathbb{R} \) then \( (a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \)

Proof: Exercise (Use induction on \( n \))

Corollary: \( \forall n \in \mathbb{N} \sum_{i=0}^{n} \binom{n}{i} = 2^n \)

Proof: Use \( a = b = 1 \) in the binomial theorem.