Ch. 8 - Functions

Highlights:
- functions
- domain and codomain
- values/ images of points
- equality of functions
- restrictions of functions
- composition of functions
- images of functions
- graphs of functions

Problem types:
- determine if something is a function (given formula, table, picture with arrows, or graph)
- determine if functions are equal
- composing functions
- finding images of functions

Notes:
- A function \( f: X \rightarrow Y \) assigns each \( x \in X \) a unique element \( y \in Y \). However, an element \( y \in Y \) may be assigned to one, many, or no elements in \( X \).
- For equality of functions, only the values matter.
- The order of composition matters!
- In a table for a function \( f: X \rightarrow Y \), each \( x \in X \) appears precisely once.
- In a picture (arrow) diagram for a function \( f: X \rightarrow Y \), each \( x \in X \) has exactly one arrow leaving it.
- A subset \( G \subseteq X \times Y \) is the graph of a function precisely when for every \( x \in X \) there is a unique \( y \in Y \) with \((x, y) \in G\).
Defn: Let $X$ and $Y$ be sets. A function (also called a map or a mapping) from $X$ to $Y$ is an assignment of a unique $y$ to each $x \in X$. If $f$ is a function from $X$ to $Y$ then we write $f: X \to Y$ and call $X$ the domain of $f$ and $Y$ the codomain of $f$. The element of $Y$ assigned to an element $x \in X$ is denoted $f(x)$ and called the value or image of $x$ under $f$. We also write $x \mapsto f(x)$.

For example, a function $f$ from $X = \{0, 1, 4, 6\}$ to $Y = \{11, 12, 20\}$ is given by the following table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>11</td>
<td>20</td>
<td>11</td>
<td>11</td>
</tr>
</tbody>
</table>

So $f(0) = 11$, $f(1) = 20$, $f(4) = 11$, $f(6) = 11$.

Note: Elements of $Y$ may occur not at all ($y = 12$), once ($y = 20$), or more than once ($y = 11$) as a value of $f$.

This can be visualized by drawing arrows from points in $X$ to points in $Y$:

![Diagram showing arrows from $X$ to $Y$]

Being a function means that every point in $X$ has precisely one edge leaving it.
Other Examples:

- identity function \( \text{id}_x : X \to X \), \( \text{id}_x(x) = x \) for all \( x \in X \)
- constant functions \( f : X \to Y \), there is \( y_0 \in Y \) so that \( f(x) = y_0 \) for all \( x \in X \)
- inclusion functions \( i : A \to X \), \( A \subseteq X \), \( i(a) = a \) for all \( a \in A \)

Defn: Two functions \( f : X \to Y \) and \( g : X' \to Y' \) are equal \( f = g \) if \( X = X' \), \( Y = Y' \), and \( \forall x \in X \) \( f(x) = g(x) \).

Example: Let \( f, g : \mathbb{R}^2 \to \mathbb{R} \) by

\[
f(x, y) = \frac{x+y}{2} + \frac{|x-y|}{2} \quad \text{and} \quad g(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } x < y \end{cases}
\]

Prove \( f = g \).

Proof: Let \( x, y \in \mathbb{R} \). We will prove by cases that \( f(x, y) = g(x, y) \).

First consider the case \( x \geq y \). Then \( |x-y| = x-y \) so

\[
f(x, y) = \frac{1}{2}(x+y+|x-y|) = \frac{1}{2}(x+y+x-y) = \frac{1}{2}(2x) = x = g(x, y).
\]

Now consider the remaining case \( x < y \). Then \( |x-y| = -(x-y) = -x+y \) so

\[
f(x, y) = \frac{1}{2}(x+y+|x-y|) = \frac{1}{2}(x+y-x+y) = \frac{1}{2}(2y) = y = g(x, y).
\]

We conclude that \( \forall (x, y) \in \mathbb{R}^2 \) \( f(x, y) = g(x, y) \) and hence \( f = g \). \( \square \)

Note: For equality, only the values of the function matter, not the method by which they are obtained.

Defn: For a function \( f : X \to Y \) and a set \( A \subseteq X \), the restriction of \( f \) to \( A \) is the function \( f|_A : A \to Y \) defined by \( f|_A(a) = f(a) \) for all \( a \in A \).
Prob: Define \( f: \mathbb{R} \to \mathbb{R} \) by \( f(x) = \frac{x^2 - 1}{x - 1} \) and define \( g: \mathbb{R} \to \mathbb{R} \) by \( g(x) = x + 1 \). Prove that \( g \mid \mathbb{R} - \{1\} = f \).

Sol: Note that \( g \mid \mathbb{R} - \{1\} \) and \( f \) have the same domain and codomain. Now consider a real number \( x \in \mathbb{R} - \{1\} \). Then
\[
f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1} = x + 1 = g(x) = (g \mid \mathbb{R} - \{1\})(x).
\]
We conclude \( g \mid \mathbb{R} - \{1\} = f \).

Defn If \( f: X \to Y \) and \( g: Y \to Z \) then their composition is the function \( g \circ f: X \to Z \) given by \( g \circ f(x) = g(f(x)) \).

Prob: Find formulas for \( f \circ g \) and \( g \circ f \) if \( f, g: \mathbb{R} \to \mathbb{R} \) are given by \( f(x) = x + 1 \) and \( g(x) = x^2 \).

Sol: \( f \circ g(x) = f(g(x)) = f(x^2) = x^2 + 1 \)
\( g \circ f(x) = g(f(x)) = g(x+1) = (x+1)^2 = x^2 + 2x + 1 \)

Note: The order of composition matters! Generally \( f \circ g \neq g \circ f \).

Defn The image of a function \( f: X \to Y \) is the set of \( y \in Y \) occurring as a value of \( f \):
\[\text{im}(f) = \{ y \in Y : \exists x \in X : f(x) = y \} \]

Prob: Define \( f: \mathbb{R} \to \mathbb{R} \) by \( f(x) = x^2 \). What is the image of \( f \)?

Sol: \( \text{im}(f) = [0, \infty) \)
Definition: The graph of a function $f: X \to Y$ is the subset of the Cartesian product $X \times Y$ given by

$$G_f = \{(x, y) \in X \times Y : f(x) = y \} = \{(x, f(x)) : x \in X\}$$

Note: A subset $G \subseteq X \times Y$ is the graph of a function if and only if for every $x \in X$ there is a unique $y \in Y$ with $(x, y) \in G$. 