Ch. 9 - Injections, Surjections, & Bijections

Highlights:
- Injection/injective/one-to-one
- Surjection/surjective/onto
- Bijection/bijective/
- Preimage
- Invertible
- Inverse

Problem types:
- Prove that a function is or is not injective, surjective, or bijective.
- Find the inverse of a function.
- Prove or disprove that two functions are inverses of each other.
**Defn:** A function \( f : X \rightarrow Y \) is:

- **an injection** (or injective or one-to-one) if
  \[ \forall x_1, x_2 \in X \quad x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \]
  (equivalently, \( \forall x_1, x_2 \in X \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \))
- **a surjection** (or surjective or onto) if
  \[ \forall y \in Y \quad \exists x \in X \quad f(x) = y \]
- **a bijection** (or bijective) if \( f \) is both
  injective and surjective (one-to-one and onto)

**Ex:** Consider \( f, g : \{a, b, c\} \rightarrow \{1, 2, 3, 4\} \)

\[
\begin{array}{c|c|c}
X & f & Y \\
\hline
a & 1 & \ast 1 \\
b & 2 & \ast 2 \\
c & 3 & \ast 3 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
X & g & Y \\
\hline
a & 1 & \ast 1 \\
b & 2 & \ast 2 \\
c & 3 & \ast 3 \\
\end{array}
\]

- \( f \) is not injective: \( f(b) = f(c) \)
- \( g \) is injective
- Neither \( f \) nor \( g \) is surjective: 4 is not a value of either function

**Prob:** Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by \( f(x) = |x| + 1 \) for all \( x \in \mathbb{R} \).
Prove that \( f \) is not surjective.

**Sol:** For any \( x \in \mathbb{R} \), \( |x| \leq 0 \) and so \( |x| + 1 \geq 1 \). Thus \( f(x) \neq 1 \)
for all \( x \in \mathbb{R} \). So 0 does not occur as a value of \( f \).
We conclude \( f \) is not surjective. \( \Box \)

**Note:** Instead of 0 we could have chosen any number less than 1, such as \( \frac{1}{2} \) or -1.
Prob: Define \( g: \mathbb{R} \to [1, \infty) \) by \( g(x) = \sqrt{x} + 1 \) for all \( x \in \mathbb{R} \). Prove that \( g \) is surjective.

Sol: Let \( y \in [1, \infty) \). Then \( y \geq 1 \) so \( y - 1 \geq 0 \) and thus \( \sqrt{y - 1} = y - 1 \). Using \( x = y - 1 \in \mathbb{R} \) we have
\[
f(x) = f(y - 1) = \sqrt{y - 1} + 1 = y - 1 + 1 = y.
\]
We have shown that \( \forall y \in [1, \infty) \exists x \in \mathbb{R} \text{ s.t. } f(x) = y. \)
We conclude \( f \) is surjective. \( \square \)

Prob: Define \( f: \mathbb{R} \to \mathbb{R} \) by \( f(x) = (3x + 1)^2 \) for all \( x \in \mathbb{R} \). Prove that \( f \) is not injective.

Sol: We have \( f(\frac{1}{3}) = (3 \cdot \frac{1}{3} + 1)^2 = 2^2 = 4 \) and \( f(-1) = (3 \cdot (-1) + 1)^2 = (-2)^2 = 4. \) Since \( \frac{1}{3} \neq -1 \) and \( f(\frac{1}{3}) = f(-1) \), we conclude that \( f \) is not injective.

Prob: Define \( g: \mathbb{Z} \to \mathbb{Z} \) by \( g(x) = (3x + 1)^2 \) for all \( x \in \mathbb{Z} \). Prove that \( g \) is injective.

Sol: \( \forall x_1, x_2 \in \mathbb{Z} \) we have
\[
g(x_1) = g(x_2) \iff (3x_1 + 1)^2 = (3x_2 + 1)^2
\]
\[
\iff (3x_1 + 1)^2 - (3x_2 + 1)^2 = 0
\]
\[
\iff ((3x_1 + 1) - (3x_2 + 1))(3x_1 + 1 + 3x_2 + 1) = 0
\]
\[
\iff 3(x_1 - x_2)(3x_1 + 3x_2 + 2) = 0
\]
\[
\iff x_1 = x_2 \text{ or } 2 = 3(-x_1 - x_2)
\]
Since \( 3 \) does not divide \( 2 \), we have \( 2 \neq 3(-x_1 - x_2) \).
Therefore \( g(x_1) = g(x_2) \iff x_1 = x_2. \) We conclude \( g \) is injective.
Defn: Let $f: X \rightarrow Y$ be a function and let $y \in Y$. An element $x \in X$ is a preimage of $y$ if $f(x) = y$.

Note: $f: X \rightarrow Y$ is:
- injective if every $y \in Y$ has at most one preimage
- surjective if every $y \in Y$ has at least one preimage
- bijective if every $y \in Y$ has exactly one preimage

Prob: Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 5x + 1$ for all $x \in \mathbb{R}$.
Prove that $f$ is a bijection.

Sol: For $x, y \in \mathbb{R}$, we have that

$x$ is a preimage of $y \iff f(x) = y 
\iff 5x + 1 = y 
\iff x = \frac{y - 1}{5}

We see from this that every $y \in \mathbb{R}$ has precisely one preimage. We conclude that $f$ is a bijection. \[ \square \]

Defn: A function $f: X \rightarrow Y$ is invertible if there exists a function $g: Y \rightarrow X$ such that

$\forall x \in X \ \forall y \in Y \ \ f(g(y)) = y \iff x = g(y)$

In this case we call $g$ an inverse of $f$ and write $g = f^{-1}$.

Theorem: A function $f: X \rightarrow Y$ is invertible if and only if it is bijective.
Proof: Let \( f: X \to Y \) be a function.

First, suppose that \( f \) is invertible. Then there is a function \( g: Y \to X \) with \( \forall x \in X \forall y \in Y \) \( f(x) = y \iff x = g(y) \).

Then for \( x \in X \) and \( y \in Y \) we have
\[
x \text{ is a preimage of } y \iff f(x) = y \iff x = g(y).
\]

This shows that every \( y \in Y \) has precisely one preimage. We conclude that \( f \) is bijective.

Next, suppose that \( f \) is bijective. Then each \( y \in Y \) has precisely one preimage. Let \( g(y) \in X \) denote the unique preimage of \( y \in Y \). Then \( g \) is a function and
\[
\forall x \in X \forall y \in Y \ f(x) = y \iff x \text{ is a preimage of } y \iff x \text{ is the unique preimage of } y.
\]

So \( g \) is an inverse of \( f \). We conclude that \( f \) is invertible. \( \Box \)

Thm: If \( f: X \to Y \) is invertible then its inverse is unique.

Proof: Suppose that \( g_1, g_2: Y \to X \) are inverses of \( f \). Consider \( y \in Y \) and set \( x_1 = g_1(y) \) and \( x_2 = g_2(y) \). Then \( f(x_1) = y \) since \( x_1 = g_1(y) \) and \( f(x_2) = y \) since \( x_2 = g_2(y) \). Therefore \( f(x_1) = f(x_2) \). By the previous theorem, we know \( f \) is injective. So we must have \( x_1 = x_2 \). Thus \( g_1(y) = x_1 = x_2 = g_2(y) \). So for every \( y \in Y \), \( g_1(y) = g_2(y) \). We conclude \( g_1 = g_2 \). \( \Box \)
Functions and subsets

Given a function $f: X \to Y$ we can define new functions:

- $\hat{f}: P(X) \to P(Y)$ by $\hat{f}(A) = \{ f(x) : x \in A \}$ for $A \in P(X)$
- $\check{f}: P(Y) \to P(X)$ by $\check{f}(B) = \{ x : x \in f^{-1}(B) \}$ for $B \in P(Y)$.

Note: It is conventional to abuse notation and write $f$ for $\hat{f}$ and $f^{-1}$ for $\check{f}$. 