1. Prove that if \( f : X \to Y \) and \( g : Y \to Z \) are bijections then \( g \circ f \) is a bijection.

**Solution:** Let \( f : X \to Y \) and \( g : Y \to Z \) be bijections.

First, we check that \( g \circ f \) is injective. Observe that \( f \) and \( g \) are both injective since they are bijective. Let \( x_1, x_2 \in X \) with \( x_1 \neq x_2 \). Since \( x_1 \neq x_2 \) and \( f \) is injective, we get that \( f(x_1) \neq f(x_2) \). Similarly, since \( f(x_1) \neq f(x_2) \) and \( g \) is injective, we get \( g(f(x_1)) \neq g(f(x_2)) \). Therefore \( g \circ f(x_1) \neq g \circ f(x_2) \). Thus \( g \circ f \) is injective.

Next, we check that \( g \circ f \) is surjective. Observe that \( f \) and \( g \) are both surjective since they are bijective. Let \( z \in Z \) (the codomain of \( g \circ f \)). Since \( g \) is surjective, there is \( y \in Y \) with \( g(y) = z \). Similarly, since \( f \) is surjective there is \( x \in X \) with \( f(x) = y \). Then \( g \circ f(x) = g(f(x)) = g(y) = z \). Thus \( g \circ f \) is surjective. We conclude that \( g \circ f \) is a bijection. \( \square \)

2. Let \( f : X \to Y \) and \( g : Y \to Z \) be functions. Prove that if \( g \circ f \) is injective then \( f \) is injective.

**Solution:** Let \( x_1, x_2 \in X \) with \( f(x_1) = f(x_2) \). Set \( y = f(x_1) = f(x_2) \). Then \( g \circ f(x_1) = g(\ f(x_1)) = g(y) = g(f(x_2)) = g \circ f(x_2) \). Since \( g \circ f \) is injective and \( g \circ f(x_1) = g \circ f(x_2) \), it must be that \( x_1 = x_2 \). So we have shown that for all \( x_1, x_2 \in X \), \( f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \). We conclude that \( f \) is injective. \( \square \)

3. Let \( f : X \to Y \) and \( g : Y \to X \) be functions. Prove that \( g \) is the inverse of \( f \) if and only if \( g \circ f = id_X \) and \( f \circ g = id_Y \).

**Solution:** First assume that \( g \) is the inverse of \( f \), meaning that for all \( x \in X \) and \( y \in Y \) we have \( f(x) = y \) precisely when \( x = g(y) \). Observe that \( g \circ f \) and \( id_X \) both have \( X \) as their domain and codomain, and \( f \circ g \) and \( id_Y \) both have \( Y \) as their domain and codomain. So we only must check that \( \forall x \in X \ g \circ f(x) = id_X(x) \) and \( \forall y \in Y \ f \circ g(y) = id_Y(y) \). Let's first consider an element \( x \in X \). Setting \( y = f(x) \), we have \( g(y) = x \) since \( g \) is the inverse of \( f \). Thus \( g \circ f(x) = g(f(x)) = g(y) = x = id_X(x) \). Thus \( g \circ f = id_X \). Next, consider an element \( y \in Y \). Setting \( x = g(y) \), we have \( f(x) = y \) since \( g \) is the inverse of \( f \). Thus \( f \circ g(y) = f(g(y)) = f(x) = y = id_Y(y) \). We conclude that \( g \circ f = id_X \) and \( f \circ g = id_Y \).

Now assume that \( g \circ f = id_X \) and \( f \circ g = id_Y \). We must show that for all \( x \in X \) and \( y \in Y \), \( f(x) = y \) if and only if \( x = g(y) \). So fix \( x \in X \) and \( y \in Y \). First assume that \( f(x) = y \). Since \( g \circ f = id_X \) we have \( x = id_X(x) = g \circ f(x) = g(f(x)) = g(y) \).

Thus \( f(x) = y \) implies \( x = g(y) \). Next assume that \( x = g(y) \). Since \( f \circ g = id_Y \) we have \( f(x) = f(g(y)) = f \circ g(y) = id_Y(y) = y \).

So \( x = g(y) \) implies \( f(x) = y \). We conclude that \( g \) is the inverse of \( f \). \( \square \)

4. Let \( f : X \to Y \) be a function with \( X \neq \emptyset \). Prove that \( f \) is injective if and only if there is a function \( g : Y \to X \) with \( g \circ f = id_X \). (Hint: for help building \( g \), review the proof that if \( f \) is bijective then \( f \) is invertible).

**Solution:** First assume that \( f \) is injective. Since \( X \neq \emptyset \) we can fix an element \( x_0 \in X \). Since \( f \) is injective, every \( y \in Y \) has at most one preimage. So we can define a function \( g : Y \to X \) by the following rule: for \( y \in Y \) we define \( g(y) \) to be the unique preimage of \( y \), if \( y \) has a preimage, and otherwise if \( y \) does not have any preimage we set \( g(y) = x_0 \). Now we check...
that $g \circ f = \text{id}_X$. Clearly $g \circ f$ and $\text{id}_X$ both have $X$ as their domain and codomain. So we only need to check that $g \circ f(x) = \text{id}_X(x)$ for all $x \in X$. Fix an element $x \in X$. Set $y = f(x)$. Then certainly $x$ is a preimage of $y$, so it is the unique preimage of $y$ and hence $g(y) = x$. Therefore $g \circ f(x) = g(f(x)) = g(y) = x = \text{id}_X(x)$. We conclude that $g \circ f = \text{id}_X$.

Now assume that there is a function $g : Y \to X$ with $g \circ f = \text{id}_X$. Let $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$. Set $y = f(x_1) = f(x_2)$. Then

$$x_1 = \text{id}_X(x_1) = g \circ f(x_1) = g(f(x_1)) = g(y) = g(f(x_2)) = g \circ f(x_2) = \text{id}_X(x_2) = x_2.$$

We have shown that for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. We conclude that $f$ is injective. \hfill $\Box$

5. Let $f : X \to Y$ and $g_1, g_2 : Y \to Z$ be functions. Assume that $g_1 \neq g_2$ and that $f$ is surjective. Prove that $g_1 \circ f \neq g_2 \circ f$.

**Solution:** Observe that $g_1$ and $g_2$ have the same domain and the same codomain. So, since $g_1 \neq g_2$, there must be some element $y \in Y$ with $g_1(y) \neq g_2(y)$. Since $f$ is surjective, there is $x \in X$ with $f(x) = y$. Then we have $g_1 \circ f(x) = g_1(f(x)) = g_1(y) \neq g_2(y) = g_2(f(x)) = g_2 \circ f(x)$. We conclude that $g_1 \circ f \neq g_2 \circ f$.

6. Let $X$ be a finite set and let $A, B \subseteq X$. Prove that if $|A| + |B| \geq |X| + 5$ then $|A \cap B| \geq 5$. (You may use the fact that if $Y \subseteq X$ then $|Y| \leq |X|$).

**Solution:** The inclusion-exclusion principle tells us that $|A \cup B| = |A| + |B| - |A \cap B|$. Since $A, B \subseteq X$ we have $A \cup B \subseteq X$, and therefore $|A \cup B| \leq |X|$. Thus $|X| \geq |A| + |B| - |A \cap B|$ and hence $|A \cap B| \geq |A| + |B| - |X| \geq 5$. \hfill $\Box$