Study Problems for Midterm 1

Note: Your exam will cover chapters 1 through 6 and will consist of four problems – one involving propositions, one involving arithmetic, one using induction, and one involving sets.

Propositions

1. Prove that “not (P or Q)” is equivalent to “(not P) and (not Q).”
2. Prove that “P ⇒ Q” is equivalent to “(P or Q) ⇒ Q.”
3. Prove that “P and (Q or R)” is equivalent to “(P and Q) or (P and R).”
4. Prove that “(P and Q) ⇒ R” is equivalent to “(P ⇒ R) or (Q ⇒ R).”
5. Prove that “(P ⇒ R) and (Q ⇒ R)” is equivalent to “(P or Q) ⇒ R.”
6. Find one statement below that is NOT equivalent to “Q or P.” Justify your answer.
   (a) not( (not Q) and (not P))
   (b) (not Q) ⇒ P
   (c) Q or (P and Q)
   (d) (Q and (not P)) or P

Arithmetic

1. Prove that if 5 divides n then 5 divides 3n + 10.
2. Prove that if x and y are distinct positive real numbers then \( \frac{x}{y} + \frac{y}{x} > 2 \).
3. Prove that if \( n^2 - 1 \) is even then n is odd.
4. Prove that there do not exist integers n and m with \( 9n - 12m = 5 \).
5. Prove that if 5 divides n then 5 does not divide 2n + 4.
6. Prove that if 5x > x^2 then 0 < x < 5.
7. Prove that if 2 divides n and 3 divides m then 6 divides nm.
8. Prove that if 9 divides 2n + m and n – m then 3 divides n.

Induction

1. Prove 5 divides \( 6^n + 4 \) for all positive integers n.
2. Prove \( \sum_{i=0}^{n}(2i + 1) = (n + 1)^2 \) for all integers \( n \geq 0 \).

3. Prove 3 divides \( n^3 - n \) for all positive integers \( n \).

4. Prove \( \sum_{i=0}^{n}2^i = 2^{n+1} - 1 \) for all integers \( n \geq 0 \).

5. Prove \( \sum_{i=0}^{n}2 \cdot 3^i = 3^{n+1} - 1 \) for all integers \( n \geq 0 \).

6. Prove \( \sum_{i=1}^{n} i(i + 1) = \frac{1}{3} n(n + 1)(n + 2) \) for all positive integers \( n \).

Sets

1. Prove \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \).

2. Prove that if \( A - B = B \) then \( B = \emptyset \).

3. Prove \( (A \cap B) - C = (A - C) \cap B \).

4. Prove that if \( (A - C) \cap B = \emptyset \) then \( A \cap B \subseteq C \).

5. Prove \( A - (B \cup C) = (A - B) \cup (A - C) \).

6. Prove or disprove: \( \mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B) \) (where \( \mathcal{P}(S) \) denotes the power set of a set \( S \))