NON-CMC SOLUTIONS OF THE EINSTEIN CONSTRAINT EQUATIONS ON ASYMPTOTICALLY EUCLIDEAN MANIFOLDS

JAMES DILTS, JIM ISENBERG, RAFE MAZZEO, AND CALEB MEIER

Abstract. In this note we prove two existence theorems for the Einstein constraint equations on asymptotically Euclidean manifolds. The first is for arbitrary mean curvature functions with restrictions on the size of the transverse-traceless data and the non-gravitational field data, while the second assumes a near-CMC condition, with no other restrictions.

1. Introduction {Intro}

This paper capitalizes on several recent advances concerning the existence of solutions of the Einstein constraint equations using the conformal method. Using these new techniques we construct solutions of these equations, either in the vacuum setting or else with coupled non-gravitational fields, on asymptotically Euclidean manifolds under two separate sets of hypotheses: either the mean curvature function τ is arbitrary but the transverse-traceless part of the data σ (and the non-gravitational field densities) are very small ("far-CMC" data) or else smallness assumptions are placed on $d\tau/\tau$ (the "near-CMC" case). We do this by adapting the methods of Holst-Nagy-Tsogterel [HNT08] and Maxwell [Ma09].

We recall that a Riemannian manifold $(M, \hat{g})$ and a symmetric 2-tensor $\hat{K}$ on M satisfy the Einstein constraint equations with non-gravitational energy-density $\hat{\rho}$ (a scalar function) and non-gravitational momentum density $\hat{J}$ (a vector field) if

$$\begin{aligned} |\hat{K}|^2_{\hat{g}} - (\mathrm{tr}_{\hat{g}}\hat{K})^2 &= R(\hat{g}) - \hat{\rho}, \\ \mathrm{div}_{\hat{g}}\,\hat{K} - \nabla \mathrm{tr}_{\hat{g}}\hat{K} &= \hat{J}. \end{aligned} \qquad (1) \quad \{vce\}$$

Note that we work here only with non-gravitational fields which require no further constraints; this is the case for fluid fields, for example. (These results easily extend to theories such as Einstein-Maxwell which do introduce extra constraints; for brevity, we do not treat such cases here.) The cosmological constant Λ is assumed to vanish because our interest here is exclusively with asymptotically Euclidean data.

One uses the conformal method to generate an initial data set $(M, \hat{g}, \hat{K}, \hat{\rho}, \hat{J})$ which satisfies the constraints (1) by first (freely) choosing the *conformal data*, which includes a Riemannian manifold (M, g), a symmetric tensor σ which is transverse ($\mathrm{div}_g \sigma = 0$) and traceless ($\mathrm{tr}_g \sigma = 0$) with respect to g, a scalar function τ (the mean curvature), a non-negative scalar function ρ, and a vector field J. One then seeks solutions ϕ (a positive scalar) and W (a vector field) of the *conformal constraint equations*

$$\begin{aligned} &(i)\ \Delta_g u - c_n R_g u + c_n|\sigma + \mathcal{D}W|^2_g u^{-N-1} - b_n \tau^2 u^{N-1} + c_n \rho u^{-\frac{N}{2}} = 0, \\ &(ii)\ \Delta_{\mathbb{L}} W + \frac{n-1}{n} u^N \nabla\tau + J = 0. \end{aligned} \qquad (2) \quad \{ce2\}$$

Key words and phrases. Einstein constraint equations, conformal method, asymptotically Euclidean manifolds.

Here R_g is the scalar curvature of g, $\mathcal{D}$ is the conformal Killing operator acting on vector fields

$$(\mathcal{D}W)_{ij} := \nabla_i W_j + \nabla_j W_i - \frac{2}{n}(\mathrm{div}_g W)\, g_{ij},$$

Δ_g is the scalar Laplacian, $\Delta_{\mathbb{L}} := -\mathrm{div}\circ\mathcal{D}$ is the vector Laplacian, and the constants N, c_n, and b_n are given by

$$N := \frac{2n}{n-2}, \quad c_n := \frac{n-2}{4(n-1)}, \quad b_n := \frac{n-2}{4n}.$$

If (ϕ, W) is a solution of (2), then the initial data set

$$\hat{g} = u^{\frac{4}{n-2}} g, \quad \hat{K} = u^{-2\frac{n+2}{n-2}}(\sigma + \mathcal{D}W)^{ij} + \frac{\tau}{n} u^{\frac{4}{n-2}} g^{ij}, \quad \hat{\rho} = u^{-\frac{3}{2}N+1}\rho, \quad \hat{J} = u^{-N} J,$$

satisfies the Einstein constraints (1).

For convenience below, if v is any positive function (in a suitable function space), then we let $W(v)$ denote the solution of equation (ii) of (2), where this function v is inserted on the right hand side. Similarly, we write the Lichnerowicz operator on the left of equation (i) of (2) as $\mathcal{N}(u, W)$. Thus a solution (u, W) of the coupled system (2) corresponds to a solution u of the single nonlocal equation $\mathcal{N}(u, W(u)) = 0$.

When τ is constant (the CMC case), these equations decouple, and it is possible to obtain incisive results for this case; see [Is95, CBIY00]. Similarly, when $\nabla\tau/\tau$ is suitably small (this is known as the near-CMC case), then many further results have been obtained using perturbation methods. The recent advances, stemming from the papers of Holst, Nagy and Tsogterel [HNT08], later refined and simplified by Maxwell [Ma09], treat the case in which τ is allowed to vary with no restrictions; these results are still perturbative in a different sense because they require σ, ρ, and J to be very small (this is a special case of the general far-CMC case). These arguments rely on the Yamabe positivity of the underlying conformal class $[g]$ and on these various smallness conditions to construct barriers. This particular far-CMC scenario has now been worked out in several settings. The original papers treat the case where M is closed; the more recent papers of Holst, Meier and Tsogtgerel [HMS13] and Dilts [Di13] treat the case where M is a manifold with boundary, considering a wide range of boundary conditions; finally, Leach [Le12] has dealt with the case where M is complete with cylindrical ends. In this paper we continue this line of research and prove an existence result in this far-CMC case for manifolds with asymptotically Euclidean ends, which is one of the standard and most important settings in relativity. The new issue to be faced here is the way that barriers must be constructed near infinity. This is similar to what must be done in the cylindrical case, but the argument here is simpler than in [Le12]. We also determine the precise asymptotics of solutions. A near-CMC result is proved here using very similar methods (we recall that [CBIY00] contains other near-CMC results for asymptotically Euclidean data sets.)

We now state our main results. The precise definitions of asymptotically Euclidean metrics, and of the weighted Sobolev spaces appearing in these statements, are all given in the next section.

{FarC} **Theorem 1.1. (Far-from-CMC)** *Suppose that (M^n, g) is a $W^{2,p}_\gamma$ asymptotically Euclidean (AE) metric with positive Yamabe invariant, where $p > n$ and $\gamma \in (2-n, 0)$, and set $\delta = \gamma/2$. Fix data $\tau \in W^{1,p}_{\delta-1}$, $\sigma \in L^\infty_{\delta-1}$, nonnegative $\rho \in L^\infty_{2\delta-2}$ and $J \in L^p_{\delta-2}$, and assume that $\|\sigma\|_{L^\infty_{\delta-1}}$, $\|\rho\|_{L^\infty_{\delta-2}}$ and $\|J\|_{L^p_{\delta-2}}$ are sufficiently small* *(depending on τ, g and n). Then there exists a solution (ϕ, W) to (2) with $W \in W^{2,p}_\delta$, $\phi > 0$ and $\phi - A_j \in W^{2,p}_\delta$ for some constant $A_j > 0$ on each end E_j of M.* {ffcmc}

Theorem 1.2. (**Near-CMC**) *Let (M, g) be AE as in Theorem 1.1, with γ, δ as above. Assume too that $\tau \in W^{1,p}_{\delta-1}$ with $\tau - Br^{2\delta-2}\|d\tau\|_{L^p_{\delta-2}} > 0$ for some $B > 0$, where r is an everywhere positive function which is the radial distance on each end of M, and that $\sigma \in L^\infty_{\delta-1}$, $\rho \in L^p_{\delta-2}$ with $\rho \geq 0$ and $J \in L^p_{\delta-2}$. Then there exists a solution (ϕ, W) to (2) with $W \in W^{2,p}_\delta$, $\phi > 0$ and $\phi - A_j \in W^{2,p}_\delta$ for some constant $A_j > 0$ on each end E_j of M.* {NearC}

The primary task in proving these theorems is to establish the existence of upper and lower barriers for equations (2). After discussing asymptotically Euclidean manifolds and function spaces in Section 2, and then reviewing the mapping properties of the scalar and vector Laplacian operators on AE manifolds in Section 3, we derive these barriers in Section 4. A standard fixed point theorem is then used in Section 5 to prove Theorems 1.1 and 1.2.

2. Asymptotically Euclidean Manifolds {AsEuc}

Let (M^n, g) be an asymptotically Euclidean (AE) manifold. This means that M is a complete manifold such that for some compact set $K \subset M$, the complement $M \setminus K$ has finitely many components, $E_1, \ldots, E_\ell$, where each E_j is diffeomorphic to the exterior of a ball in a Euclidean space, $E_j \cong \mathbb{R}^n \setminus B_R(0)$, and on each of these ends, the metric g is asymptotic to the Euclidean metric. More precisely, recall that a function $u \in W^{k,p}_\delta(\mathbb{R}^n)$ if

$$\sum_{|\beta| \leq k} ||r^{-\delta - \frac{n}{p} + |\beta|} \partial^\beta u||_{L^p} < \infty.$$

Here r is a smooth positive function on M which agrees with the radial function $|x|$ on each end. To extend these spaces and norms to tensors, as needed in the characterization of the decay of the metric g above, we require this regularity and decay for each component with respect to a constant frame in the background Euclidean metric. Thus we say that g is AE of class $W^{k,p}_\gamma$, for some $\gamma < 0$, if in a fixed Euclidean coordinate system for that end,

$$g|_{E_j} - g_{\mathrm{Euc}} \in W^{k,p}_\gamma.$$

The regularity of the tensor field $\hat{K}$ and the scalar and vector fields $\hat{\rho}$ and $\hat{J}$ are defined analogously. We refer the reader to [Ba86] for a survey of the well-known properties of these spaces.

We single out one fact which we used repeatedly: if $p > n$, and $w \in W^{1,p}_\delta$ for any $\delta \in \mathbb{R}$, then

$$|w| \leq r^\delta ||w||_{W^{1,p}_\delta}. \tag{3}$$ {weightest}

The initial data set $(M, \hat{g}, \hat{K}, \Lambda, \hat{\rho}, \hat{J})$ is said to be asymptotically Euclidean if $\hat{g} - g_{\mathrm{Euc}} \in W^{k,p}_\gamma$, $\hat{K} \in W^{k-1,p}_{\gamma-1}$, and $\rho, J \in W^{k-2,p}_{\gamma-2}$ for some $\gamma < 0$.

In the following, we always assume that (M, g) is AE of class $W^{2,p}_\gamma$ with $p > n$ and $2(2-n) < \gamma < 0$, but omit writing this explicitly. We also always assume that $\delta = \gamma/2$, so $2 - n < \delta < 0$. All results below have obvious modifications if we assume that g is AE of class $W^{k,p}_\gamma$ with $k > 1 + n/p$.

3. Mapping properties of the scalar and vector Laplacians {Map}

The mapping properties of elliptic operators on asymptotically Euclidean spaces is now classical, going back at least to [Mc79], but see also [Ma05] and the appendix in [?]. We record a few such results needed below, pertaining to the solvability of the inhomogeneous linear equation

$$Pu = f,$$

where P is either the conformal Laplacian $\Delta_g - c_n R$ or else the vector Laplacian $\Delta_{\mathbb{L}}$.

{veclapfred}

Proposition 3.1. *If (M, g) is AE, then*

{fredvL}
$$P : W^{2,p}_\delta \longrightarrow L^p_{\delta-2} \tag{4}$$

is Fredholm of index zero, and there is an a priori estimate: there is a constant $C > 0$ such that

$$\|\psi\|_{\psi^{2,p}_\delta} \leq C \left(\|P\psi\|_{L^p_{\delta-2}} + \|\psi\|_{L^\infty} \right)$$

for all $\psi \in W^{2,p}_\delta$. The map (4) is an isomorphism if and only if P has no nullspace in $W^{2,p}_\delta$. For $P = \Delta - c_n R$, this is the case provided the Yamabe invariant $\mathcal{Y}([g])$ is positive; while for $P = \Delta_{\mathbb{L}}$, this holds if (M, g) admits no global conformal Killing fields. Under this isomorphism condition, the a priori estimate above can be strengthened to

{injbound}
$$\|\psi\|_{W^{2,p}_\delta} \leq C \|P\psi\|_{L^p_{\delta-2}}. \tag{5}$$

We record two useful corollaries.

{samas}

Proposition 3.2. *If P is the conformal Laplacian $\Delta - c_n R$ and $R \geq 0$, and if $f = r^{\gamma-2} + \hat{f}$, where $\hat{f} \in L^p_{\gamma'-2}$ for $\gamma' < \gamma$, then there is a unique solution w to $Pw = f$ with $w = c_\gamma r^\gamma + \hat{w}$, $c_\gamma = (\gamma^2 + (n-2)\gamma)^{-1}$, and $\hat{w} \in W^{2,p}_{\gamma''}$ where $\gamma'' = \max\{\gamma', 2\gamma\}$ if this number is greater than $2 - n$ (or else $\gamma'' \in (2 - n, \gamma)$).*

Proof. Write $w = c_\gamma r^\gamma + \hat{w}$ and let $\bar{g}$ be a $W^{2,p}$ metric which agrees with g away from the ends but is exactly Euclidean on each E_j. Then we must solve

$$(\Delta - c_n R)\hat{w} = \hat{f} - c_\gamma(\Delta_{\bar{g}} - R_{\bar{g}})r^\gamma - c_\gamma((\Delta - c_n R) - (\Delta_{\bar{g}} - c_n R_{\bar{g}}))r^\gamma.$$

The second term on the right is L^p with compact support, while the third term lies in $L^p_{2\gamma-2}$. Using the nonnegativity of R (to rule out the kernel), the result follows from Proposition 3.1. □

{boundDW}

Proposition 3.3. *If (M, g) is AE and has no conformal Killing fields, and if $f \in L^p_{\delta-2}$, then the unique solution $W \in W^{2,p}_\delta$ to $\Delta_{\mathbb{L}} W = f$ satisfies*

{DWest}
$$\|\mathcal{D}W\|_\infty \leq C r^{\delta-1} ||f||_{L^p_{\delta-2}}. \tag{6}$$

Proof. Combining (5) and (3), we get

$$r^{1-\delta}|\mathcal{D}W| \leq \|\mathcal{D}W\|_{L^\infty_{\delta-1}} \leq C\|\mathcal{D}W\|_{W^{1,p}_{\delta-1}} \leq C\|W\|_{W^{2,p}_\delta} \leq C||f||_{L^p_{\delta-2}},$$

and this gives (6). □

4. Barriers

{sec:barriers}

We begin by recalling the notion of *global sub- and supersolutions*. The function ϕ_+ is called a global supersolution for (2) if $\mathcal{N}(\phi_+, W(\phi)) \le 0$ whenever $0 < \phi \le \phi_+$. Similarly, ϕ_- is called a global subsolution if $\mathcal{N}(\phi_-, W(\phi)) \ge 0$ whenever $\phi \in L^p$ and $\phi_- \le \phi$.

Theorem 4.1. (Far-from-CMC Global Supersolution) *Let (M, g) be AE with positive Yamabe invariant; i.e., $\mathcal{Y}([g]) > 0$. If $||\sigma||_{L^\infty_{\delta-1}}$, $||J||_{L^p_{\delta-2}}$ and $||\rho||_{L^\infty_{2\delta-2}}$ are sufficiently small, then there exists a global supersolution $\phi_+ > 0$ with $\phi_+ - \eta \in W^{2,p}_\gamma$ for some constant $\eta > 0$.* {farcmcbarrier}

Proof. Choose a smooth, positive function F which equals $r^{\gamma-2}$ outside a compact set (recall that γ indexes the asymptotic behavior of the AE metric). By Proposition 3.1, there exists a (unique) $\Psi = c_\gamma r^\gamma + \hat{\Psi}$, with $\hat{\Psi} \in W^{2,p}_{2\gamma}$ such that

$$(\Delta - c_n R)\Psi = -F + c_n R, \tag{7}$$ {Psi}

or equivalently

$$(\Delta - c_n R)(1 + \Psi) = -F. \tag{8}$$

Note that, by the maximum principle, $1 + \Psi > 0$.

Now set $\phi_+ = \eta(\Psi + 1)$, where the constant $\eta > 0$ is to be chosen below. We claim that, for appropriate η, ϕ_+ is a global supersolution. To verify this, we first note that from (6), with $f = \frac{n-1}{n}\phi^N \nabla\tau + J$, we have

$$\|\mathcal{D}W\|_\infty \le C r^{\delta-1} \left(\|d\tau\|_{L^p_{\delta-2}} \|\phi\|^N_\infty + \|J\|_{L^p_{\delta-2}} \right), \tag{9}$$ {DWestim}

and hence

$$|\sigma + \mathcal{D}W|^2 \le C r^{2\delta-2} (\|d\tau\|^2_{L^p_{\delta-2}} \|\phi\|^{2N}_\infty + ||\sigma||^2_{L^\infty_{\delta-1}} + \|J\|^2_{L^p_{\delta-2}}).$$

Since Ψ decays at the precise rate r^γ (and is strictly positive), then deleting subscripts denoting the norms for simplicity, we calculate

$$\mathcal{N}(\phi_+, W(\phi)) \le \\ -\eta F + r^{2\delta-2} \left(C_1 \eta^{N-1} + C_2 \eta^{-N-1}(||\sigma||^2 + ||J||^2) + C_3 \eta^{-\frac{N}{2}} ||\rho|| \right).$$

The constants C_1, C_2 and C_3 depend only on F and the dimension n. Since $2\delta - 2 = \gamma - 2 < 0$ and $N - 1 > 1$, we first choose η sufficiently small so that

$$-\frac{1}{2}\eta F + C_1 \eta^{N-1} r^{2\delta-2} < 0,$$

and then choose $||\sigma||$, $||J||$ and $||\rho||$ sufficiently small (depending on C_1, F, n and η), so that

$$-\frac{1}{2}\eta F + r^{2\delta-2} \left(C_2 \eta^{-N-1}(||\sigma||^2 + ||J||^2) + C_3 \eta^{-\frac{N}{2}} ||\rho|| \right) < 0$$

as well. This proves that ϕ_+ is a global supersolution. □

Theorem 4.2. (Near-CMC Global Super-Solution) *Let (M, g) be AE with $\mathcal{Y}([g]) > 0$, and fix any $\rho \in L^p_{2\delta-2}$ with $\rho \ge 0$, $J \in L^p_{\delta-2}$ and $\sigma \in L^\infty_{\delta-1}$. Suppose that $\tau \in W^{1,p}_{\delta-1}$ satisfies $\tau - B r^{2\delta-2} \|d\tau\|_{L^p_{\delta-2}} > 0$ for some constant B depending only on the dimension n and the constant appearing in (6). Then there exists a global supersolution for (2).* {nearcmcgss}

Proof. We first claim that we can choose $u \in W^{2,p}_{\gamma}$ such that

$$(\Delta - c_n R)(1+u) - b_n \tau^2 (1+u)^{N-1} = 0.$$

This prescribed scalar curvature problem has a solution by [CBIY00, Sec VII] and $1+u>0$ by the maximum principle. Next define $v \in W^{2,p}_{2\delta}$ by

$$\nabla((1+u)^2 \nabla v) - b_n \tau^2 (1+v) = -c_n(\rho + |\sigma|^2);$$

its existence and uniqueness is guaranteed by Proposition 3.1, and as before, $1+v>0$. (Strictly speaking, we have only stated that result for $P = \Delta - c_n R$, but the proof applies equally well to this operator.) Now set $\phi_+ = \eta u v$, where the constant η is chosen below. We calculate that

$$u(\Delta - c_n R)\phi_+ = \eta(-c_n\rho - c_n|\sigma|^2 + b_n\tau^2 v + b_n \tau^2 u^N v),$$

so for any $0 < \phi < \phi_+$,

$$\begin{aligned} u\mathcal{N}(\phi_+, W(\phi)) &= \eta(-c_n\rho - c_n|\sigma|^2 + b_n\tau^2 v + b_n\tau^2 u^N v) \\ &\quad - (\eta v)^{N-1} b_n \tau^2 u^N + c_n|\sigma + \mathcal{D}W|^2 u^{-N}(\eta v)^{-N-1} + c_n(\eta v)^{-\frac{N}{2}}\rho u^{-\frac{N-2}{2}} \\ &\le b_n\tau^2(\eta u^N v + \eta v - (\eta v)^{N-1}u^N) - c_n\eta(\rho + |\sigma|^2) \\ &\quad + 2c_n(|\sigma|^2 + |\mathcal{D}W|^2)(\eta v)^{-N-1}u^{-N} + c_n \eta^{-\frac{N}{2}} \rho u^{-\frac{N-2}{2}} v^{-\frac{N}{2}}. \end{aligned}$$

By (9) and the inequality $\phi < \phi_+$, we have

$$\begin{aligned} |\mathcal{D}W|^2 &\le C r^{2\delta-2}((\sup\phi)^{2N}||d\tau||_{L^p_{\delta-1}} + ||J||_{L^p_{\delta-2}})^2 \\ &\le C' r^{2\delta-2}((\eta u v)^{2N}||d\tau||^2_{L^p_{\delta-1}} + ||J||^2_{L^p_{\delta-2}}), \end{aligned}$$

and this leads to the estimate

$$\begin{aligned} u\mathcal{N}(\phi_+, W(\phi)) \;\le\;& c_n|\sigma|^2\left(-\eta + 2\eta^{-N-1}v^{-N-1}u^{-N}\right) \\ +\;& c_n\rho\left(-\eta + \eta^{-\frac{N}{2}}v^{-\frac{N}{2}}u^{\frac{-N-2}{2}}\right) \\ +\;& \eta^{N-1}\left(-\frac{b_n}{3}\tau^2 + C_1||d\tau||^2_{L^p_{\delta-1}} r^{2\delta-2}\right)v^{N-1}u^N \\ +\;& \left(-\frac{b_n}{3}\eta^{N-1}\tau^2 v^{N-1}u^N + \eta v b_n\tau^2 + \eta v b_n \tau^2 u^N\right) \\ +\;& \left(-\frac{b_n}{3}\eta^{N-1}\tau^2 v^{N-1}u^N + C_2\eta^{-N-1}v^{-N-1}u^{-N}r^{2\delta-2}||J||_{L^p_{\delta-2}}\right). \end{aligned}$$

All five terms here can be made negative. Indeed, the constant C_1 in the third term depends on the constant in (6), so we can apply the near-CMC assumption hypothesis here; the other terms are negative so long as η is sufficiently large. □

We now turn to the construction of a global subsolution. This turns out to be the same for both the far-CMC and near-CMC cases.

{gss} **Theorem 4.3. (Global Subsolution)** *Let $(M, g, \sigma, \tau, \rho, J)$ be a set of conformal data satisfying the hypotheses of either Theorem 1.1 or Theorem 1.2. Let $\psi \in W^{2,p}_{\delta}$ be chosen so that $1+\psi > 0$ and $\tilde{g} = (1+\psi)^{N-2}g$ has scalar curvature $R_{\tilde{g}} = -\frac{n-1}{n}\tau^2$ (see* [CBIY00, Sec VII] *). Then $\alpha(1+\psi)$ is a global subsolution for any $0 < \alpha \le 1$.*

Proof. With this definition of ψ, let $\phi_- = \alpha(1+\psi)$. Then

$$\mathcal{N}(\phi_-, W(\phi)) = b_n\tau^2(1+\psi)^{N-1}(\alpha - \alpha^{N-1}) + |\sigma + \mathcal{D}W(\phi)|^2(\alpha(1+\psi))^{-N-1} + c_n\rho(\alpha(1+\psi))^{-\frac{N}{2}} \geq 0,$$

as required. Note that this does not even require that $\phi \geq \phi_-$. □

Since $\phi_- \to \alpha$ and $\phi_+ \to \eta$ at infinity, and both are strictly positive, we can choose α sufficiently small so that $\phi_- < \phi_+$ everywhere.

5. Fixed point Theorem and proof of the main results

{sec:FixedPoint

Just as for the analogous far-CMC results on closed manifolds [HNT08] and [Ma09], once the existence of global sub- and supersolutions has been established, then the existence of a solution (ϕ, W) to (2) is obtained using the Schauder fixed point theorem. Since this proof is quite similar to the one for closed manifolds, we only sketch it here.

Theorem 5.1. *For AE conformal data sets satisfying the hypotheses of Theorem 1.1 or Theorem 1.2, there exists a solution (ϕ, W) to (2), with $\phi_- \leq \phi \leq \phi_+$. Moreover, on each end E_j of M, $\phi - A_j \in W^{2,p}_\gamma$ for some constant A_j on each end E_j of M.*

Proof. Let $\mathcal{C}^0_+$ denote the set of strictly positive bounded functions on M. If $\phi \in \mathcal{C}^0_+$, then by Proposition 3.1, the vector field $W(\phi) \in W^{2,p}_\delta$ is well-defined. Next, let $T(W)$ be the solution ϕ to $\mathcal{N}(\phi, W) = 0$ for any $W \in W^{2,p}_\delta$. This map is also well-defined. We claim that any $\phi = T(W)$ can be decomposed as $\phi = A_j + \hat{\phi}$ on each end of M, where $\hat{\phi} \in W^{2,p}_\delta$. (Thus if we let A be a smooth function which equals A_j on each end, then $\phi = A + \hat{\phi}$.) Granting this for the moment, let S denote the compact inclusion $\mathbb{R} \oplus W^{2,p}_\delta \hookrightarrow \mathcal{C}^0$. A solution (ϕ, W) to (2) corresponds to a fixed point of the mapping $Q = S \circ T \circ W$. The continuity of W and S are obvious, while the continuity of T follows from the implicit function theorem. Up to the claim about the decomposition of ϕ stated above, this proves that Q is a continuous compact mapping.

Define the bounded convex set $\mathcal{S} := \{\phi \in \mathcal{C}^0_+ : \phi_- \leq \phi \leq \phi_+\}$. By construction, Q maps $\mathcal{S}$ to itself, and hence $Q(\mathcal{S})$ is relatively compact. Denote by $\mathcal{H}$ its closed convex hull. Thus $\mathcal{H} \subset \mathcal{S}$, and $Q : \mathcal{H} \to \mathcal{H}$. By the Schauder fixed point theorem, $\mathcal{H}$ contains a fixed point ϕ of Q. Standard estimates imply that ϕ and $W(\phi)$ both have the desired regularity.

The proof is finished once we prove that $T(W) = \phi = A + \hat{\phi}$, as claimed earlier. For this we rewrite the Lichnerowicz equation as $\Delta u = f \in L^p_{\delta-2}$, where f incorporates a number of terms involving u. Since $\Delta : W^{2,p}_\delta \to L^p_{\delta-2}$ is an isomorphism, there exists a function $\hat{u} \in W^{2,p}_\delta$ such that $\Delta\hat{u}$ is equal to this same function f, hence $w = u - \hat{u}$ is a bounded harmonic function on M. It is well known that on a manifold with asymptotically Euclidean ends, any such function tends to a constant on each end. Furthermore, given any constants A_j, there is a bounded harmonic function which tends to A_j on E_j. Indeed, define $A = \sum \chi_j A_j$ as above, where each χ_j is a cutoff function which equals 1 on the end E_j and vanishes elsewhere. Then $\Delta A \in L^p_{\delta-2}$, so there exists a function $\hat{A} \in W^{2,p}_\delta$ such that $\Delta\hat{A} = \Delta A$, so $A - \hat{A}$ is the bounded harmonic function in question. □

Acknowledgments. All authors are partially supported by the NSF FRG grant DMS-1263431. As well, RM is partially supported by NSF DMS-1105050, and JI by PHY-1306441. We thank MSRI for support during the period in which this research was carried out. The authors thank Michael Holst and David Maxwell for helpful conversations.

References

[Ba86] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math, **39** (1986), 661–693.

[CBIY00] Y. Choquet-Bruhat, J. Isenberg, and J. W. York, Jr, *Einstein constraints on asymptotically Euclidean manifolds*, Phys. Rev. D **61** (2000), 1–20.

[Di13] J. Dilts, *The Einstein Constraint Equations on Compact Manifolds with Boundary*, Available as arXiv:1310.2303 [gr-qc].

[HMS13] M. Holst, C. Meier, and G. Tsogtgerel, *Non-CMC Solutions to the Einstein Constraints with Apparent Horizon Boundaries*, Available as arXiv:1310.2302 [gr-qc].

[HNT08] M. Holst, G. Nagy, and G. Tsogtgerel, *Rough Solutions to the Einstein Constraint Equations on closed manifolds without near-CMC conditions.*, Comm. Math Phys., **288 (2)** (2009), 547–613.

[Is95] J. Isenberg, *Constant mean curvature solutions of the Einstein constraint equations on closed manifolds*, Classical Quantum Gravity **12** (1995), 2249–2274.

[Le12] J. Leach, *A far-from-CMC existence result for the constraint equations on manifolds with ends of cylindrical type*, To appear, Classical Quantum Gravity. Available as arXiv:1306.0608 [gr-qc].

[Ma05] D. Maxwell, *Solutions of the Einstein Constraints equations with apparent horizon boundaries*, Comm. Math Phys., **253 (3)** (2005), 561–583.

[Ma09] D. Maxwell, *A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature*, Math Res. Lett., **16 (4)** (2009), 627–645.

[Mc79] R. C. McOwen, *The behavior of the Laplacian on weighted Sobolev spaces*, Comm. Pure Appl. Math. **32** (1979), 783–795.

University of Oregon
E-mail address: jdilts@uoregon.edu

University of Oregon
E-mail address: jisenberg@uoregon.edu

Stanford University
E-mail address: mazzeo@math.stanford.edu

UC San Diego
E-mail address: c1meier@math.ucsd.edu