Homework

- Calculate and compare the eigenvalues and combinatorial eigenfunctions for $\tilde{G} = C_{2n}$ and $G = P_{n+1}$.
- Relax the distance-regular condition for $\tilde{G}$ in Theorem A and $\Gamma$ in Theorem B as much as possible.

1. Eigenvalues of Distance Regular Coverings

Recall that for graphs $G$ and $H$ with second-smallest eigenvalues $\lambda_G$ and $\lambda_H$,

- If $H$ is a contraction of $G$, then $\lambda_G \leq \lambda_H$.
- If $G$ is a covering of $H$, then all eigenvalues of $H$ are eigenvalues of $G$. Moreover, an eigenvalue of $G$ is an eigenvalue of $H$ if the projection of its combinatorial eigenfunction is not identically 0.

Example 1. Consider the cycle $C_4$, which is a covering of the two-fold path $P_3$.

\[ 
\begin{array}{c}
 1 \\
 2 \\
 3 \\
 4 \\
\end{array} 
\quad \xrightarrow{\text{projection}} \quad 
\begin{array}{c}
 1 \\
 2 \\
 3 \\
 4 \\
\end{array} 
\quad \xrightarrow{\text{projection}} 
\begin{array}{c}
 a \\
 b \\
 c \\
\end{array} 
\]

Say we have the projection $\pi(1) = a, \pi(2) = b, \pi(3) = b$ and $\pi(4) = c$.

$C_4$ has eigenvalues $\lambda_0 = 0, \lambda_1 = 1, \lambda_2 = 1$, and $\lambda_3 = 2$. Let $f_1, f_2, f_3, f_4$ be the corresponding combinatorial eigenfunctions. Recall that the projection $\pi f_i$ maps a vertex to the average of $f_i$ on its preimage under $\pi$.

Then we have
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\[
\begin{array}{c|c|c|c|c|c|c}
  f_0 & \pi f_0 & f_1 & \pi f_1 & f_2 & \pi f_2 & f_3 \\
  \downarrow 1 \Rightarrow 1 & a \Rightarrow 1 & \downarrow 1 \Rightarrow 1 & a \Rightarrow 1 & \downarrow 1 \Rightarrow 0 & a \Rightarrow 0 & \downarrow 1 \Rightarrow 1 \\
  2 \Downarrow 1 & b \Rightarrow 1 & 2 \Rightarrow 0 & b \Rightarrow 0 & 2 \Rightarrow 1 & b \Rightarrow 0 & 2 \Rightarrow -1 \\
  3 \Downarrow 1 & c \Rightarrow 1 & 3 \Rightarrow 0 & c \Rightarrow -1 & 3 \Rightarrow -1 & c \Rightarrow 0 & 3 \Rightarrow -1 \\
  4 \Downarrow 1 & 4 \Rightarrow -1 & 4 \Rightarrow 0 & 4 \Rightarrow 1 & & & \\
\end{array}
\]

Since only \( \pi f_2 \) is identically 0, we have \( \lambda_0 = 0, \lambda_1 = 1, \) and \( \lambda_3 = 2 \) as eigenvalues of the twofold \( P_{n+1} \) as well.

**Example 2.** The Petersen graph is a covering of a multi-path \( P_3 \) with edge weights 3 and 6 and a loop of weight 12 on the terminal vertex. Thus, we may use the eigenvalues of this multi-path to determine three of the eigenvalues of the Petersen graph.

Both examples 1 and 2 concern examples of a special class of graphs known as distance-regular graphs. A distance-regular graph is a graph with the following property: given any two vertices \( u \) and \( v \) with distance \( k \) between them, then the number of vertices \( w \) at distance \( i \) from \( u \) and distance \( j \) from \( v \) depends solely on \( i, j \), and \( k \) and not on \( u \) and \( v \). One nice class of distance-regular graphs are distance-transitive graphs: given any four vertices \( u, v, w \) and \( x \) such that the distance from \( u \) to \( v \) equals the distance from \( w \) to \( x \), there exists a graph automorphism sending \( u \) to \( w \) and \( v \) to \( x \).

Note that for a distance regular graph \( G \), if we take \( u \) to be a vertex achieving the diameter, \( v \) to be its neighbor on such a path, \( k = 1, i = \text{diam}(G), \) and \( j = \text{diam}(G) - 1 \), this implies that every vertex in the graph achieves the diameter.

**Theorem A.** Suppose \( \tilde{G} \) is a distance-regular covering of a weighted rooted path \( G \) with projection \( \pi \). Suppose that \( G \) has \( k+1 \) vertices, where \( k = \text{diam}(G) \). Then \( \tilde{G} \) has exactly \( k+1 \) distinct eigenvalues.

_Proof._ Let \( \mathcal{L} \) be the normalized Laplacian of the path \( G \). Since \( G \) is a path, it must be the case that \( I, \mathcal{L}, \mathcal{L}^2, \ldots, \mathcal{L}^k \) are all linearly independent. This implies that the minimal polynomial of \( \mathcal{L} \) has degree at least \( k+1 \). By the Cayley-Hamilton theorem, the minimal polynomial of \( \mathcal{L} \) has degree at most \( k+1 \). We conclude that \( G \) has exactly \( k+1 \) distinct eigenvalues.

Since each eigenvalue of \( G \) is an eigenvalue of \( \tilde{G} \), this means that \( \tilde{G} \) has at least \( k+1 \) distinct eigenvalues. We aim to show that every other eigenvalue \( \lambda \) of \( \tilde{G} \) must also be an eigenvalue of \( G \). So let \( f \) be a combinatorial eigenfunction of \( \tilde{G} \) associated with \( \lambda \). There exists some vertex \( v_0 \) such that \( f(v_0) \neq 0 \). Since \( \tilde{G} \) is distance-regular, we may consider \( \pi(v_0) \) to be the root of the path \( G \). It follows that \( \pi f(\pi(v_0)) = f(v_0) \neq 0 \). Hence, \( \pi f \) is not identically 0, and \( \lambda \) must belong to the spectrum of \( G \).

We may also recover the multiplicities of these eigenvalues simply from the weighted path.
**Theorem B.** If $\Gamma$ on $n$ vertices is a distance-regular covering of a weighted rooted path $P$ with diameter $k$, and if $P$ has eigenvalues $\{\lambda_i\}_{i=1}^s$, then the multiplicity $m$ of $\lambda_i$ in $\Gamma$ is

$$m(\lambda_i) = \frac{ng_i^2(v_0)}{|g_i|^2},$$

where $g_i$ is an eigenfunction of $P$ associated with $\lambda_i$, and $v_0$ is the root of $P$.

**Proof.** For $j$ from 0 to $k$, define the matrices

$$A_j(x, y) = \begin{cases} 1 & \text{if dist}(x, y) = j \text{ in } \Gamma \\ 0 & \text{otherwise} \end{cases}.$$ 

Notice $A_1 = A$. Let $v_j$ be the $j$th vertex on path $P$, and let $V_j = \pi^{-1}(v_j)$. We claim that

$$\sigma(A_j) = \{ |V_j| \frac{f(v_j)}{f(v_0)} \mid f \text{ is a combinatorial eigenfunction of } P \}.$$ 

To see this, first note that each $A_j$ is a polynomial in the matrix $A$, so each $A_j$ shares the same eigenfunctions.

For combinatorial eigenfunction $f : V(P) \to \mathbb{R}$, we may talk about $f$ on $V(\Gamma)$ by defining $f(x) = f(\pi(x))$.

Suppose that for some $\lambda'$ and $f$ we have

$$A_j f = \lambda' f.$$ 

Notice that $P$ has adjacency eigenvalue 0 with eigenfunction $\varphi_0 = (1, 0, \ldots, 0)$. So on the one hand,

$$\langle \varphi_0, \pi A_j f \rangle = \langle \varphi_0, \pi \lambda' f \rangle \\
= \lambda' \langle \varphi_0, \pi f \rangle \\
= \lambda' f(v_0).$$

But also,

$$\langle \varphi_0, \pi A_j f \rangle = \varphi_0^T \pi A_j f \\
= (\varphi_0^T \pi A_j) f \\
= (|V_j| e_j^T) f \quad \text{(where } e_j \text{ is the standard basis vector)} \\
= |V_j| f(v_j).$$

Equating $\lambda' f(v_0)$ and $|V_j| f(v_j)$ proves the claim.

Now suppose $f_i$ is the $i$th combinatorial eigenfunction of $P$ associated with eigenfunction $\lambda_i$. Consider the matrix

$$M_i = \sum_{j=0}^k f_i(v_j) A_j.$$
Since $A_0 = I$ is the only $A_j$ with diagonal entries, we have $\text{Tr}(M_i) = n f_i(v_0)$.

We know the eigenvalues of $A_j$ are $|V_j| \frac{f_p(v_j)}{f_p(v_0)}$ with multiplicity $m(\lambda_p)$, so also

$$\text{Tr}(M_i) = \sum_{j=0}^{k} f_i(v_j) \text{Tr}(A_j)$$

$$= \sum_{j=0}^{k} f_i(v_j) \sum_{p=0}^{s} m(\lambda_p) \frac{f_p(v_j)}{f_p(v_0)} |V_j|$$

$$= \sum_{j=0}^{k} m(\lambda_i) \frac{f_i^2(v_j)}{f_i(v_0)} |V_j|$$

$$= m(\lambda_i) \frac{1}{f_i(v_0)} \sum_{j=0}^{k} f_i^2(v_j) |V_j|,$$

where the second to last equation comes from orthogonality of the eigenfunctions $f_p$ and $f_i$ when $p \neq i$.

Equating the two results for $\text{Tr}(M_i)$ and solving for $m(\lambda_i)$ gives

$$m(\lambda_i) = \frac{n f_i(v_0)^2}{\sum_{j=0}^{k} f_i^2(v_j) |V_j|} = \frac{ng_i^2(v_0)}{||g_i||^2},$$

where $g_i$ is the actual eigenfunction corresponding to $f_i$. \hfill \square

Theorems A and B have great computational value. When we have a distance-regular covering of a weighted path with size $n$ and diameter $k$, we reduce the complexity of finding its eigenvalues (even with multiplicities) from a polynomial in $n$ to a polynomial in $k < n$.

**Example 3.** To imagine the power of these theorems, consider the infinite $k$-tree $T_k$ (which is distance-regular) and the weighted rooted path $P$ whose weight from vertex $v_i$ to vertex $v_{i+1}$ is $k(k-1)^i$. $T_k$ can give us a covering of $P$, which in turn we can use to find eigenvalues of $T_k$ (if we are careful with applying the theorems to an infinite graph).