My favorite application using eigenvalues: quantum random walks on graphs

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Abstract

State transfer on graphs has potential applications to quantum computing and communication. We present history, background, and a few results on state transfer of quantum random walks on graphs. Basic mathematical formulations and applications of eigenvalues of a graph’s adjacency matrix relevant to this subject are introduced without dwelling on physical concepts from which they originate.

1 History and motivation

In classical computation, it is well known that randomized algorithms are in certain cases capable of providing considerable speed increases when compared to deterministic algorithms. Analogously, in the realm of quantum computation, random walks of quantum particles on graphs have been shown to provide exponential speed increases over classical algorithms for certain problems. Although quantum computation is not yet technologically realizable, the famous algorithms of Shor ([8]) and Grover ([7]) already demonstrated in the 1990s that quantum algorithms can be used to solve efficiently problems thought to be “hard” in the realm of classical computation.

More recently, Bose ([2]) demonstrated how this model can be applied to communicate quantum states over a network, and Childs ([3]) showed how quantum random walks can be used to provide a model of universal quantum computation. To this end, it is useful to understand how state transfer on graphs operates.

Over the last several years, researchers have investigated properties of quantum random walks on certain families of graphs. Among these is perfect
state transfer, a property of quantum random walks not shared by classical random walks: under certain conditions, a quantum state will be transmitted to another state on the graph with 100% probability. The works of Bašić and Petković ([1]), Christandl et al. ([4]), as well as Godsil ([5],[6]) give partial characterizations of graphs with this property, showing that this phenomenon is intimately linked with the spectrum of a graph’s adjacency matrix.

2 Setup

Let $G$ be a (possibly weighted) graph on $n$ vertices with adjacency matrix $A$. Then define $H(t) = \exp(itA)$. Given some starting superposition of states $v_0 \in \mathbb{C}^n$ satisfying $v_0^*v_0 = 1$ and standard basis vector $e_i$ corresponding to the vertex (state) $v_i \in V(G)$, we interpret $|e_i^*v_0|$ as the probability of measuring state $v_i$ at time $t = 0$. The quantum random walk is defined by $\psi(t) = H(t)v_0$, which gives a new unit-length superposition of states, and again $|e_i^*\psi(t)|$ is interpreted as the probability of measuring state $v_i$ at time $t$.

We say that a graph exhibits perfect state transfer from vertex $v_i$ to $v_j$ if there exists some time $\tau$ such that $|e_j^*H(\tau)e_i| = 1$. For a simple example, consider the walk on $K_2$ with starting state $v_0 = e_1$. In this case it is easy to compute $H(t)$ as

$$H(t) = \begin{pmatrix} \cos(t) & i\sin(t) \\ i\sin(t) & \cos(t) \end{pmatrix}$$

Since $|e_2^*H(\pi/2)e_1| = 1$, we see that perfect state transfer is achieved between the two vertices of the graph. In this case the classical random walk behaves similarly; this is not in general true (The quantum random walk on $P_3$, for example, also exhibits perfect state transfer between antipodal vertices.).

A graph exhibits periodicity if it has perfect state transfer from a vertex to itself. It turns out that every graph with perfect state transfer has periodicity as a direct consequence. A graph is periodic if it exhibits periodicity at each of its vertices.
3 Results

Below we present a handful of results from the literature on periodicity and perfect state transfer.

Theorem 1 (Christandl et al. [4]). The unweighted path $P_n$ does not have perfect state transfer for $n > 3$. However, it is possible to weight the edges of $P_n$ (with weights suggested by the hypercube graph) to allow antipodal perfect state transfer for arbitrary $n$.

Theorem 2 (Godsil [6]). A graph $G$ is periodic if and only if either:

(a) $G$ has integer eigenvalues, or
(b) the eigenvalues of $G$ are rational multiples of $\sqrt{\Delta}$ for some square-free integer $\Delta$.

In the latter case $G$ is bipartite.

Theorem 3 (Godsil [6]). Suppose $G$ is a connected vertex-transitive graph with vertices $u$ and $v$, and perfect state transfer from $u$ to $v$ occurs at time $\tau$. Then $H(\tau)$ is a scalar multiple of a permutation matrix with order two and no fixed points, and it lies in the centre of the automorphism group of $G$.

Theorem 4 (Bašić and Petković [1]). The only unitary Cayley graphs that have perfect state transfer are $K_2$ and $C_4$.

4 Method

The proofs of the above results rely largely on examining the spectra of the relevant graphs. The proof of the first follows from looking at the eigenvalues and eigenvectors of paths which can be computed directly, while the others require a bit more subtlety.

In particular, consider symmetric matrix (for our purposes the adjacency matrix) $A$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$, and let $E_j$ be the orthogonal projection matrix onto the eigenspace associated with $\lambda_j$. Then we have a
convenient representation of \( H(t) \), namely

\[
H(t) = \sum_{i=1}^{m} e^{it\lambda_i} E_i.
\]

This formulation leads to an observation proved in Godsil [5] that if \( \theta_a, \theta_b, \theta_c, \theta_d \) are eigenvalues of \( A \) such that \( e_i \) for some periodic vertex is nonzero when projected onto all four eigenspaces, then \( (\theta_a - \theta_b)/(\theta_c - \theta_d) \in \mathbb{Q} \). From this the first theorem of Godsil quickly follows. The result of Bašić and Petković similarly follows by looking at divisibility conditions involving eigenvalues.

The second theorem of Godsil is proved using coherent algebras.

References


