Turn off and put away your cell phone.  
You may not use any notes or calculators during this exam.  
Read each question carefully, and answer each question completely.  
Show all of your work; no credit will be given for unsupported answers.  
Write your solutions clearly and legibly; no credit will be given for illegible solutions.  
If any question is not clear, ask for clarification.
1. (25 points) Let \( A = \begin{bmatrix} 9 & 3 & 3 \\ 3 & 5 & 7 \\ 3 & 7 & 11 \end{bmatrix} \) and \( b = \begin{bmatrix} 24 \\ 34 \\ 50 \end{bmatrix} \). Solve \( Ax = b \) with Cholesky Decomposition, forward substitution, and backward substitution. i.e. First find \( R \) such that \( R^T R = A \), second solve \( R^T y = b \) for \( y \), and third solve \( Rx = y \) for \( x \).

**Answer:**

\[
R = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{then} \quad y = \begin{bmatrix} 8 \\ 13 \\ 3 \end{bmatrix} \quad \text{then} \quad x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\]
2a. (20 points). Given the points (-4,-3), (-1,0), (2,0), (3,1), (5,2), calculate the equation of the line that best approximates this data. i.e. Find the least squares line, \( p(t) = x_1 + x_2t \), that minimizes the sum of the squares of the residuals, \( \sum_{i=1}^{5} |y_i - p(t_i)|^2 \), where the given data is in the form \((t_i, y_i)\).

**ANSWER:**

The equation of the line is \( p(t) = x_1 + x_2t \) where \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) is the least squares solution to \( Ax = b \) with

\[
A = \begin{bmatrix} 1 & -4 \\ 1 & -1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.
\]

Compute \( x \) by solving the normal equations \( A^T Ax = A^T b \). \( A^T A = \begin{bmatrix} 5 & 5 \\ 5 & 55 \end{bmatrix} \) and \( A^T b = \begin{bmatrix} 0 \\ 25 \end{bmatrix} \). Therefore \( x = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \) and \( p(t) = -\frac{1}{2} + \frac{1}{2}t \).

2b. (5 points). Write the overdetermined system of equations in the form \( Ax = b \) but do not solve them used to find the equation of the parabola, \( p(t) = x_1 + x_2t + x_3t^2 \), that best approximates the data in Question 2a.

**ANSWER:**

\[
Ax = b \quad \text{with} \quad A = \begin{bmatrix} 1 & -4 & 16 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\]
3a. (10 point) Let $A$ be nonsingular, and let $x$ and $\hat{x} = x + \delta x$ be the solutions of $Ax = b$ and $A\hat{x} = b + \delta b$ respectively. Prove $\frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|}$.

**ANSWER:**

$Ax = b$ then $\|Ax\| = \|b\|$ and from the definition of an induced norm, we have $\|A\|\|x\| \geq \|b\|$. Inverting this, we get (i) $\frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}$.

From $A(x + \delta x) = b + \delta b$, subtract $Ax = b$ to get $\delta x = A^{-1} \delta b$.

Then (ii) $\|\delta x\| \leq \|A^{-1}\| \|\delta b\|$. Multiplying (i) and (ii), we get $\frac{\|\delta x\|}{\|x\|} \leq K(A) \frac{\|\delta b\|}{\|b\|}$.

3b. (5 points). Let $A = \begin{bmatrix} 100 & 99.01 \\ 100 & 99 \end{bmatrix}$. Calculate $\kappa_1(A)$.

**ANSWER:**

$A^{-1} = \begin{bmatrix} -99 & 99.01 \\ 100 & -100 \end{bmatrix}$ and $\kappa_1(A) = \|A\| \|A^{-1}\| = 200 \cdot 199.01 = 39802$

3c. (10 points). Let $A = \begin{bmatrix} 100 & 99.01 \\ 100 & 99 \end{bmatrix}$. Find $b, \delta b, x,$ and $\delta x$ such that $Ax = b$, $A(x + \delta x) = b + \delta b$, $\frac{\|\delta b\|}{\|b\|}$ is small, $\frac{\|\delta x\|}{\|x\|}$ is large, and $\frac{\|\delta x\|}{\|x\|} = \kappa_1(A) \frac{\|\delta b\|}{\|b\|}$. Note: this uses the same $A$ as Question 3b.

**ANSWER:**

In order for $\frac{\|\delta x\|}{\|x\|} = \kappa_1(A) \frac{\|\delta b\|}{\|b\|}$, we need $\|A\| \|x\| = \|b\|$ and $\|A^{-1}\| \|\delta b\| = \|\delta x\| = \|A^{-1} \delta b\|$.

Therefore we need $x$ in the direction of maximum magnification of $A$ (with regard to the 1-norm) and we need $\delta b$ in the direction of maximum magnification of $A^{-1}$ (with regard to the 1-norm). Let $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then $b = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$. Let $\delta b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $\delta x = \begin{bmatrix} 99.01 \\ -100 \end{bmatrix}$.
4. (25 point) Calculate a permutation matrix $P$, a unit lower triangle matrix $L$ with $|l_{i,j}| \leq 1$ for all $i$ and $j$, and an upper triangle matrix $U$ such that $PA = LU$ given $A = \begin{bmatrix} 2 & 5 & 6 & 8 \\ 4 & 8 & 16 & 12 \\ 1 & 4 & 8 & 9 \\ 1 & 3 & -6 & 0 \end{bmatrix}$.

**ANSWER:**

\[
P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix}, \quad \text{and } U = \begin{bmatrix} 4 & 8 & 16 & 12 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & -12 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]
5a. (10 point) Richardson’s Method uses the splitting $A = M - N$ with $M = \frac{1}{\omega} I$. Calculate the convergence rate of Richardson’s Method applied to $Ax = b$ with $A = \begin{bmatrix} 6 & 1 \\ 1 & 4 \end{bmatrix}$ using $\omega = \frac{1}{6}$?

**ANSWER:**

$$M = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}, \quad N = M - A = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}, \quad G = M^{-1}N = \begin{bmatrix} 0 & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}.$$  

We calculate the eigenvalues of $G$ with $\det(G - \lambda I) = \lambda^2 - \frac{1}{3} \lambda - \frac{1}{36} = 0$ 

$$\lambda = \frac{1 \pm \sqrt{2}}{6}$$ and the convergence rate equals $\rho(G) = \frac{1 + \sqrt{2}}{6}$. 

5b. (10 points). When solving $Ax = b$ with $A$ from Question 5a, does Richardson’s Method converge using $\omega = \frac{1}{3}$? Why or why not?

**ANSWER:**

$$M = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad N = M - A = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}, \quad G = M^{-1}N = \begin{bmatrix} -1 & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$  

We calculate the eigenvalues of $G$ with $\det(G - \lambda I) = \lambda^2 + \frac{4}{3} \lambda + \frac{1}{3} = 0$ 

$$\lambda = \frac{-2 \pm \sqrt{2}}{3}$$ and Richardson’s Method does NOT converge since $\rho(G) = \frac{-2 - \sqrt{2}}{3} > 1$.

5c. (5 points). Prove the error relationship $e^{(k+1)} = (M^{-1}N) e^{(k)}$ for all iterative methods defined by $Mx^{(k+1)} = Nx^{(k)} + b$ used to solve $Ax = b$ where $A = M - N$ and $e^{(k)} = x - x^{(k)}$.

**ANSWER:**

$Ax = b$ and $A = M - N$ imply $(M - N)x = b$ and $Mx = Nx + b$. Subtract this from $Mx^{(k+1)} = Nx^{(k)} + b$, we get $M \left(x^{(k+1)} - x\right) = N \left(x^{(k)} - x\right)$. Then $e^{(k+1)} = (M^{-1}N) e^{(k)}$. 


6a. (25 points) Write a function in Matlab that takes as input the number of iterations $k$, an $n$ by $n$ matrix $A$, and the number $n$. Have this function run the Power Method for $k$ iterations on an initial guess of the vector of all 1's and output the dominant eigenvalue and its corresponding eigenvector. Use only basic programming. (i.e. You can not use Matlab’s ability to multiply a matrix by a matrix and you can not use Matlab’s ability to multiply a matrix by a vector.)

**ANSWER:**

```matlab
function [q, s] = PowerMethod (k, A, n)
    for t = 1:k
        qnew = zeros(n,1);
        for i = 1:n
            for j = 1:n
                qnew(i) = qnew(i) + A(i,j) * q(j);
            end
            qnew(i) = qnew(i) + A(i,j) * q(j);
        end
        s = 0;
        for i = 1:n
            if abs(qnew(i)) > abs(s)
                s = qnew(i);
            end
        end
        q = qnew / s;
    end
end
```
7a. (10 points) Prove \(\|x\|_2 \leq \|x\|_1\) and \(\|x\|_1 \leq \sqrt{n}\|x\|_2\) for all vectors \(x \in \mathbb{R}^n\).

**ANSWER:**

\[
\|x\|_2^2 = \sum_{k=1}^n |x_k|^2 \leq \left( \sum_{k=1}^n |x_k|^2 \right)^2 = \|x\|_2^4 \quad \text{implies that} \quad \|x\|_2 \leq \|x\|_1.
\]

Given \(x \in \mathbb{R}^n\), let \(y \in \mathbb{R}^n\) such that \(y_i = \begin{cases} 1 & \text{if } x_i \geq 0 \\ -1 & \text{if } x_i < 0 \end{cases}\) then

\[
\|x\|_1 = x^T y \leq \|x\|_1, \quad \|y\|_2 = \sqrt{n} \|x\|_2.
\]

The middle inequality follows from the Cauchy-Schwarz inequality.

7b. (5 points) Find a vector \(x \in \mathbb{R}^3\) such that \(\|x\|_\infty = \frac{1}{2} \|x\|_1\).

**ANSWER:**

Many answers exist. One example is \(x = \left[1 \quad \frac{1}{2} \quad \frac{1}{2}\right]^T\).

Theorem 5.4.20b: Suppose \(A \in \mathbb{R}^{n \times n}\) is symmetric positive definite then \(A\) has \(n\) orthonormal eigenvectors, \(v_1, v_2, \ldots, v_n\), and \(n\) positive real eigenvalues, \(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0\).

7c. (10 points) Suppose \(A \in \mathbb{R}^{n \times n}\) is symmetric positive definite, prove \(\|A\|_2 = \rho(A)\). Hints: show \(\|A\|_2 \geq \rho(A)\) and show \(\|A\|_2 \leq \rho(A)\). Remember \(\rho(A) = \max_k \left|\lambda_k\right|\). Use Theorem 5.4.20b.

**ANSWER:**

Given s.p.d. \(A \in \mathbb{R}^{n \times n}\), let \(v_1, v_2, \ldots, v_n\) and \(\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n > 0\) be its real eigenvectors and corresponding real eigenvalues with \(v_j^T v_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}\).

\[
\|A\|_2 = \max_{x \neq 0} \left\{ \frac{\|Ax\|_2}{\|x\|_2} \right\} = \frac{\|Av_1\|_2}{\|v_1\|_2} = \frac{\|Av_2\|_2}{\|v_2\|_2} = \cdots = \frac{\|Av_n\|_2}{\|v_n\|_2} = \|\lambda_1\| = \rho(A)
\]

Given \(x \in \mathbb{R}^n\), \(x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n\) for some \(c_1, c_2, \ldots, c_n\) since \(v_1, v_2, \ldots, v_n\) are a basis of \(\mathbb{R}^n\).

\[
\|A\|_2 \leq \max_{x \neq 0} \left\{ \frac{\|A(c_1 v_1 + c_2 v_2 + \ldots + c_n v_n)\|_2}{\|c_1 v_1 + c_2 v_2 + \ldots + c_n v_n\|_2} \right\} = \max_{x \neq 0} \left\{ \frac{\|c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \ldots + c_n \lambda_n v_n\|_2}{\|c_1 v_1 + c_2 v_2 + \ldots + c_n v_n\|_2} \right\} = \max_{x \neq 0} \left\{ \|c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \ldots + c_n \lambda_n v_n\|_2 \right\} = \|\lambda_1\| = \rho(A)
\]

Therefore \(\|A\|_2 = \rho(A)\).
8a. (10 points) \( Ax = b \) with \( A = \begin{bmatrix} 7 & 2 & 1 \\ 1 & 8 & 1 \\ 2 & 2 & 9 \end{bmatrix} \) and \( b = \begin{bmatrix} 17 \\ 27 \\ 28 \end{bmatrix} \). Starting with guess \( x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \), calculate one iteration of Gauss-Seidel Method.

**ANSWER:**

\[
x^{(1)} = \begin{bmatrix} \frac{1}{2}(17 - 2 - 1) \\ \frac{1}{8}(27 - 1 \cdot (2) - 1) \\ \frac{1}{8}(28 - 2 \cdot (2) - 2 \cdot (3)) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}
\]

8b. (5 points) Algorithm 8.2.19 defines one iteration of the SOR Method. The algorithm begins with vector variable \( x = x^{(k)} \) and ends with \( x = x^{(k+1)} \). (It uses temporary scalar variables \( \hat{x} \) and \( \delta \).)

Let \( A = \begin{bmatrix} 7 & 2 & 1 \\ 1 & 8 & 1 \\ 2 & 2 & 9 \end{bmatrix} \), \( b = \begin{bmatrix} -3 \\ 49 \end{bmatrix} \), \( x^{(5)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \), and SOR \( \omega = 1.5 \). Use Algorithm 8.2.19 to compute \( x^{(6)} \).

**Algorithm 8.2.19:**

\[
x \leftarrow x^{(k)}
\]

for \( i = 1, ..., n \)

\[
\hat{x} \leftarrow \frac{1}{a_{ii}} \left( b_i - \sum_{j \neq i} a_{ij} x_j \right)
\]

\[
\delta \leftarrow \hat{x} - x_i
\]

\[
x_i \leftarrow x_i + \omega \delta
\]

end

**ANSWER:**

\[
\begin{align*}
\hat{x}_1 &= \frac{1}{2}(24 - 2 - 1) = 3, \quad \delta_1 = 3 - 1 = 2 \\
\hat{x}_2 &= \frac{1}{8}(-3 - 1 \cdot (4) - 1) = -1, \quad \delta_2 = -1 - 1 = -2 \\
\hat{x}_3 &= \frac{1}{8}(49 - 2 \cdot (4) - 2 \cdot (-2)) = 5, \quad \delta_3 = 5 - 1 = 4 \\
\end{align*}
\]

\[
x^{(6)} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}
\]

8c. (10 points) In terms of \( n \), how many flops are needed to complete one iteration of the Gauss-Seidel method? Assume \( Ax = b \) is being solved where \( A \) is an \( n \) by \( n \) matrix.

**ANSWER:**

Each element of the new vector \( x^{(k+1)} \) requires 1 division, \( n-1 \) multiplications, and \( n-1 \) subtractions.

There are \( n \) elements to compute, therefore flops = \( n(1 + (n-1) + (n-1)) = 2n^2 - n \).