Math 20D  
Final Exam  
December 10, 2015

Turn off and put away your cell phone.  
You may not use any calculators during this exam.  
You are allowed one handwritten 8.5” by 11” cheat sheet double sided.  
Read each question carefully, and answer each question completely.  
Show all of your work; no credit will be given for unsupported answers.  
Write your solutions clearly and legibly; no credit will be given for illegible solutions.  
If any question is not clear, ask for clarification.

ANSWERS

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1. (15 points) Find the solution of the initial value problem \( x' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} x \), \( x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \).

ANSWER:

Characteristic equation is \( r^2 + 6r + 9 = 0 \).

Therefore the eigenvalues are \( r = -3, -3 \) and the only eigenvector is \( \xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

Solve \( (A - rI)\eta = \xi \) for \( \eta \). The first equation says \( \eta_1 - \eta_2 = \frac{1}{4} \).

Therefore \( \eta = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} \). If we let \( \alpha = 0 \), then \( \eta = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} \).

The general solution is \( x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} \).

Plug in the initial conditions,
\( \begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} \) implies \( c_1 = 2 \) and \( c_2 = 4 \).

So, \( x = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-3t} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} t e^{-3t} \).
2a. (8 points) The system \( x' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} x \) has general solution \( x = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} \). Draw a phase portrait for this system. (A phase portrait is a plot in the \( x_1, x_2 \)-plane that shows a representative sample of trajectories for a given system.)

**ANSWER:**

![Phase portrait](image)

2b. (2 points) In the solution from problem 2a, is the origin a saddle point, a node, or a spiral point? Is it asymptotically stable or unstable?

**ANSWER:**

Unstable saddle point.
3. (15 points) Use the Laplace Transform to solve the initial value problem \( y'' + 2y' + 5y = e^{-t}, \ y(0) = \frac{1}{4}, \ y'(0) = \frac{1}{2}. \) (Helpful Laplace Transforms are at the bottom of this page.)

**ANSWER:**

The Laplace Transform of the ODE is

\[
s^2F - \frac{1}{2}s - \frac{1}{4} + 2(sF - \frac{1}{4}) + 5F = \frac{1}{s+1}
\]

\[
F = \frac{1}{(s+1)(s^2 + 2s + 5)} + \frac{\frac{1}{2}s + 1}{s^2 + 2s + 5}.
\]

Expanding the first fraction using partial fractions, we get

\[
F = \frac{\frac{1}{4}}{s+1} + \frac{-\frac{1}{4}s - \frac{1}{4}}{s^2 + 2s + 5} + \frac{\frac{1}{2}s + 1}{s^2 + 2s + 5} = \frac{\frac{1}{4}}{s+1} + \frac{\frac{3}{4}}{(s+1)(s^2 + 2s + 5)} = \frac{1}{4} \left( \frac{1}{s+1} \right) + \frac{3}{8} \left( \frac{2}{((s+1)^2 + 2^2)} \right)
\]

Therefore

\[
y = \frac{1}{4} e^{-t} + \frac{3}{8} e^{-t} \sin 2t
\]

From Table 6.2.1: \( L\{1\} = \frac{1}{s}, \ L\{e^{at}\} = \frac{1}{s-a}, \ L\{t^n\} = \frac{n!}{s^{n+1}}, \ L\{\sin at\} = \frac{a}{s^2 + a^2}, \ L\{\cos at\} = \frac{s}{s^2 + a^2}, \ L\{e^{bt}\} = \frac{b}{(s-a)^2 + b^2}, \ L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2 + b^2}, \ L\{e^{at}\sin bt\} = \frac{n!}{(s-a)^{n+1}} \)
4. (15 points) Seek power series solutions of the differential equation \( y'' - xy = 0 \) about \( x_0 = 0 \). Find the recurrence relation. Find the first four terms in each of the two solutions \( y_1 \) and \( y_2 \) (unless the series terminates sooner).

**Answer:**

Substitute \( y = \sum_{n=0}^{\infty} a_n x^n \) into the ODE and get
\[
\sum_{n=2}^{\infty} a_n (n-1)x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0
\]

Move \( x \) inside,
\[
\sum_{n=2}^{\infty} a_n (n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0
\]

Match exponents on \( x \),
\[
\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0
\]

Take one term out of first summation to match bottom indexes
\[
2a_0 + \sum_{n=1}^{\infty} (a_{n+2} (n+2)(n+1) - a_{n-1}) x^n = 0
\]

Combine summations,
\[
2a_0 + \sum_{n=1}^{\infty} \left( a_{n+2} (n+2)(n+1) - a_{n-1} \right) x^n = 0
\]

Matching \( x \) terms implies that \( a_2 = 0 \) and \( a_{n+2} (n+2)(n+1) - a_{n-1} = 0 \)

For \( n = 0, 1, 2, \ldots \) we have the recurrence relation \( a_{n+2} = \frac{a_n}{(n+3)(n+2)} \).

Using this relation, we find
\[
y_1 = 1 + \frac{1}{(3)(2)} x^3 + \frac{1}{(6)(5)(4)(3)} x^6 + \ldots
\]
\[
y_2 = x + \frac{1}{(4)(3)} x^4 + \frac{1}{(7)(6)(5)(4)(3)} x^7 + \ldots
\]

And \( y = a_0 y_1 + a_1 y_2 \)
5. (15 points) A tank originally contains 100 gal of fresh water. Then water containing \( \frac{1}{2} \) lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the well-stirred mixture is allowed to leave at the same rate. Find the amount of salt in the tank after \( t = 50 \ln 2 \approx 34.657 \) minutes.

**ANSWER:**
The ODE modeling this process is

“change in salt per time” = \( \frac{dQ}{dt} = \frac{1}{2}(2) - \frac{Q}{100} = “salt in” \) minus “salt out”.

Solving, we get \( Q(t) = 50 + ce^{-\frac{t}{50}} \). The initial condition \( Q(0) = 0 \) implies that \( c = -50 \).

So \( Q(t) = 50 - 50e^{-\frac{t}{50}} \) and substituting in \( t = 50 \ln 2 \), we get \( Q(50 \ln 2) = 50 - 50e^{-\ln^2} = 25 \).
6. (15 points) Use the method of reduction of order to find the general solution of the differential equation \( t^2 y'' + 3ty' + y = 0, \ t > 0, \ y_1(t) = t^{-1}. \)

**ANSWER:**

\( y_2(t) = u(t)y_1(t), \) we need to solve for \( u(t). \)

\( y_2 = ut^{-1}, \ y_2' = u't^{-1} - ut^{-2}, \ y_2'' = u''t^{-1} - u't^{-2} - u't^{-2} + 2ut^{-3} = u''t^{-1} - 2u't^{-2} + 2ut^{-3} \)

Substituting into the ODE, we get \( (u''t - 2u' + 2u^{-1}) + (3u' - 3ut^{-1}) + ut^{-1} = 0 \) thus \( u'' + u' = 0 \)

Let \( w = u', \) then \( w't + w = 0, \) thus \( \frac{d}{dt}(wt) = 0 \) and \( wt = k_1. \)

Therefore \( u' = k_1t^{-1}. \) Thus \( u = k_1 \ln t + k_2 \) and \( y_2 = k_1t^{-1} \ln t + k_2t^{-1} \)

The general solution is \( y = c_1t^{-1} + c_2t^{-1} \ln t \)
7. (15 points) Use the method of diagonalization to find the general solution of the differential equation

\[ x' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t. \]

**ANSWER**

The characteristic equation is \( r^2 - 4r + 3 = 0 \). Thus \( r = 1, 3 \) with corresponding \( \xi = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

Substituting \( x = Ty \) with \( T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \) into our original ODE of \( x' = Ax + g \), we get \( Ty' = ATy + g \).

Multiplying by \( T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \), we get \( y' = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} y + T^{-1} g \) with \( T^{-1} g = \begin{bmatrix} 0 \\ t \end{bmatrix} \).

So \( y_1' = y_1 \) and solving, we get \( y_1 = c_1 e^{t} \).

And \( y_2' = 3y_2 + t \) and solving we get \( y_2 e^{-3t} = \int t e^{-3t} \), thus \( y_2 = c_2 e^{3t} - \frac{t}{3} - \frac{1}{9} \).

Since \( x = Ty \), we have

\[ x = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} t - \frac{1}{9} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

Alternatively, you will get the same answer if you use

\[ T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \] and \( T^{-1} = T^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \).