2.1.21. Consider the IVP: \( y' - \frac{y}{2} = 2 \cos(t) \) with \( y(0) = a \).

a. The direction field for the given differential equation is given in Figure 1 below. There is an initial value \( a_0 \approx -1 \) that makes the solution oscillates about zero. This is indicated by the red path in the same figure. As \( t \) becomes large, if \( a > a_0 \) then the solution goes to \( +\infty \), if \( a < a_0 \) then the solution goes to \( -\infty \).

b. We have

\[
y' - \frac{y}{2} = 2 \cos(t) \Rightarrow e^{-t/2}y' - \frac{e^{-t/2}}{2}y = 2e^{-t/2} \cos(t)
\]

\[
\Rightarrow \frac{d}{dt}(ye^{-t/2}) = 2e^{-t/2} \cos(t)
\]

\[
\Rightarrow ye^{-t/2} = \int 2e^{-t/2} \cos(t) dt = -\frac{4}{5}e^{-t/2}(\cos(t) - 2 \sin(t)) + c
\]

\[
\Rightarrow y(t) = -\frac{4}{5} \cos(t) + \frac{8}{5} \sin(t) + ce^{t/2}.
\]

Suppose \( y(0) = a \), then \( a = y(0) = c - \frac{4}{5} \Rightarrow c = a + \frac{4}{5} \). Thus, the solution to the IVP is

\[
y(t) = -\frac{4}{5} \cos(t) + \frac{8}{5} \sin(t) + \left(a + \frac{4}{5}\right)e^{t/2}.
\]

This also shows that the critical value \( a_0 \) is \( -\frac{4}{5} \).
c. The solution from part (b) confirms our observation of the behavior described in (a).

2.1.31. Consider the IVP \( y' - \frac{3}{2}y = 3t + 2e^t \) with \( y(0) = y_0 \).

\[
y' - \frac{3}{2}y = 3t + 2e^t \quad \Rightarrow \quad e^{-3t/2}y' - \frac{3}{2}e^{-3t/2}y = 3te^{-3t/2} + 2e^{-t/2}
\]

\[
\Rightarrow \quad \frac{d}{dt}(ye^{-3t/2}) = 3te^{-3t/2} + 2e^{-t/2}
\]

\[
\Rightarrow \quad ye^{-3t/2} = \int \left( 3te^{-3t/2} + 2e^{-t/2} \right) dt = -\frac{2}{3}e^{-3t/2} (2 + 6e^t + 3t) + c
\]

\[
\Rightarrow \quad y(t) = -\frac{4}{3} - 2t - 4e^t + ce^{3t/2}.
\]

Under the initial condition, \( y_0 = y(0) = -\frac{4}{3} - 4 + c \Rightarrow c = y_0 + \frac{16}{3} \).

Thus, the solution to this IVP is

\[
y(t) = -\frac{4}{3} - 2t - 4e^t + \left( y_0 + \frac{16}{3} \right)e^{3t/2}.
\]

The critical value is \( y_0 = -\frac{16}{3} \). Under the initial condition \( y(0) = y_0 = -\frac{16}{3} \), \( y(t) \to -\infty \) as \( t \to \infty \).

2.2.1. For \( y \neq 0 \), we have

\[
\frac{dy}{dx} = \frac{x^2}{y} \quad \Rightarrow \quad ydy = x^2dx
\]

\[
\Rightarrow \quad \int ydy = \int x^2dx
\]

\[
\Rightarrow \quad \frac{y^2}{2} = \frac{x^3}{3} + c
\]

\[
\Rightarrow \quad 3y^2 - 2x^3 = c.
\]

2.2.2. For \( y \neq 0 \) and \( x \neq -1 \), we have

\[
\frac{dy}{dx} = \frac{x^2}{y(1 + x^3)} \quad \Rightarrow \quad ydy = \frac{x^2}{1 + x^3} dx
\]

\[
\Rightarrow \quad \int ydy = \int \frac{x^2}{1 + x^3} dx
\]

\[
\Rightarrow \quad \frac{y^2}{2} = \frac{1}{3} \ln |1 + x^3| + c
\]

\[
\Rightarrow \quad y^2 - \frac{2}{3} \ln |1 + x^3| = c.
\]

2.2.9. Given the IVP \( y' = (1 - 2x)y^2 \) with \( y(0) = -\frac{1}{6} \).
a. Observe that $y \equiv 0$ is not a solution of this IVP. When $y \neq 0$, we have

$$\frac{dy}{dx} = (1 - 2x)y^2 \Rightarrow y^{-2}dy = (1 - 2x)dx$$

$$\Rightarrow \int y^{-2}dy = \int (1 - 2x)dx$$

$$\Rightarrow -\frac{1}{y} = x - x^2 + c$$

$$\Rightarrow y(x) = \frac{1}{x^2 - x - c}.$$  

Under the initial condition, $-\frac{1}{6} = y(0) = -\frac{1}{c} \Rightarrow c = 6.$

Thus, the solution to the given IVP is

$$y(x) = \frac{1}{x^2 - x - 6}.$$  

b. The graph of the solution is given in Figure 2 below. Observe that

$$\lim_{x \to -2^-} y(x) = -\infty \text{ and } \lim_{x \to 3^+} y(x) = -\infty.$$  

![Figure 2](image)

Figure 2: The solution to the IVP $y' = (1 - 2x)y^2$ with $y(0) = -\frac{1}{6}$.  

c. The solution is defined on the interval $-2 < x < 3$. 

2.2.15. Given the IVP $y' = \frac{2x}{1 + 2y}$ with $y(2) = 0$.

a. When $y \neq -\frac{1}{2}$, we have

$$\frac{dy}{dx} = \frac{2x}{1 + 2y} \Rightarrow (1 + 2y)dy = 2xdx$$

$$\Rightarrow \int (1 + 2y)dy = \int 2xdx$$

$$\Rightarrow y + y^2 = x^2 + c$$

$$\Rightarrow y(x) = \frac{-1 \pm \sqrt{4x^2 + 1 + 4c}}{2} = \frac{-1 \pm \sqrt{4x^2 + K}}{2}. \quad (1)$$
Under the initial condition, we obtain

\[ 0 = y(2) = \frac{-1 \pm \sqrt{16 + K}}{2} \Rightarrow K = -15. \]

This also means that only the function \( y \) with positive sign in (1) can be a solution. Thus, the solution to the given IVP is

\[ y(x) = \frac{-1 + \sqrt{4x^2 - 15}}{2}. \]

b. The graph of the solution is given in Figure 3 below.

![Figure 3: The solution to the IVP \( y' = \frac{2x}{1+2y} \) with \( y(2) = 0 \).]

Figure 3: The solution to the IVP \( y' = \frac{2x}{1+2y} \) with \( y(2) = 0 \).

c. The square root \( \sqrt{4x^2 - 15} \) is defined for either \( x > \frac{\sqrt{15}}{2} \) or \( x < -\frac{\sqrt{15}}{2} \). However, the initial condition \( y(2) = 0 \) limits our solution to only the positive side. Hence, the solution is defined on the interval \( x > \frac{\sqrt{15}}{2} \).

2.2.23. Consider the IVP \( y' = 2y^2 + 2xy^2 = y^2(2 + x) \) with \( y(0) = 1 \). Observe that \( y \equiv 0 \) is not a solution of this IVP. When \( y \neq 0 \), we have

\[
\frac{dy}{dx} = y^2(2 + x) \Rightarrow y^{-2}dy = (2 + x)dx
\]

\[ \Rightarrow \int y^{-2}dy = \int (2 + x)dx \]

\[ \Rightarrow -\frac{1}{y} = 2x + \frac{x^2}{2} + c \]

\[ \Rightarrow y(x) = -\frac{2}{x^2 + 4x + c}. \]

Under the initial condition, \( 1 = y(0) = -\frac{2}{c} \Rightarrow c = -2. \)
Thus, the solution to the given IVP is
\[ y(x) = -\frac{2}{x^2 + 4x - 2}. \]

Observe that
\[ y(x) = -\frac{2}{x^2 + 4x - 2} = \frac{2}{6 - (x + 2)^2} \geq \frac{1}{3}. \]

Thus, the minimum of the solution occurs at \( x = -2 \), this minimum value is \( \frac{1}{3} \).

2.2.30. Given the differential equation
\[ \frac{dy}{dx} = \frac{y - 4x}{x - y}. \]

a. Dividing the top and bottom of the RHS by \( x \neq 0 \) yields
\[ \frac{dy}{dx} = \frac{y - 4x}{x - y} = \frac{(y/x) - 4}{1 - (y/x)}. \]

b. Let \( v = \frac{y}{x} \) or \( v(x) = xv(x) \) so that \( \frac{dy}{dx} = v + x \frac{dv}{dx} \).

c. Therefore,
\[ v + x \frac{dv}{dx} = \frac{dy}{dx} = \frac{y - 4x}{x - y} = \frac{v - 4}{1 - v} \Rightarrow x \frac{dv}{dx} = \frac{v - 4}{1 - v} - v = \frac{v^2 - 4}{1 - v} \Rightarrow \frac{1 - v}{v^2 - 4} dv = \frac{1}{x} \, dx \]
which is a separable differential equation.

d. Integrate the two sides of (2) gives
\[ \int \frac{1 - v}{v^2 - 4} \, dv = \int \frac{1}{x} \, dx \Rightarrow \int \left( -\frac{3/4}{v - 2} + \frac{1/4}{v + 2} \right) \, dv = \int \frac{1}{x} \, dx \]
\[ \Rightarrow -\frac{3}{4} \ln |v - 2| - \frac{1}{4} \ln |v + 2| = \ln |x| + c \]
\[ \Rightarrow \ln |(v - 2)(v + 2)^3| = \ln |x^{-4}| + c \]
\[ \Rightarrow (v - 2)(v + 2)^3 = Cx^{-4}. \]

(3)
e. Lastly, by replacing \( v \) with \( y/x \) in (3) we obtain the solution to the given equation as
\[ \left( \frac{y}{x} - 2 \right) \left( \frac{y}{x} + 2 \right)^3 = \frac{C}{x^4} \Rightarrow (y - 2x)(y + 2x)^3 = C. \]
2.2.33. Assuming all the expressions are well-defined, using the technique from Exercise 2.2.30, we have:

\[
\frac{dy}{dx} = \frac{4y - 3x}{2x - y} \Rightarrow \frac{dy}{dx} = \frac{4(y/x) - 3}{2 - (y/x)} \text{ for } x \neq 0
\]

\[
\Rightarrow v + x \frac{dv}{dx} = \frac{4v - 3}{2 - v} \text{ where } y(x) = xv(x)
\]

\[
\Rightarrow x \frac{dv}{dx} = \frac{4v - 3}{2 - v} - v = \frac{(v - 1)(v + 3)}{2 - v}
\]

\[
\Rightarrow \frac{2 - v}{(v - 1)(v + 3)} dv = \frac{1}{x} dx
\]

\[
\Rightarrow \int \frac{2 - v}{(v - 1)(v + 3)} dv = \int \frac{1}{x} dx
\]

\[
\Rightarrow -\frac{5}{4} \ln |v + 3| + \frac{1}{4} \ln |v - 1| = \ln |x| + c
\]

\[
\Rightarrow \ln \left| \frac{v - 1}{(v + 3)^5} \right| = \ln |x^4| + c \quad (4)
\]

\[
\Rightarrow \frac{v - 1}{(v + 3)^5} = Cx^4
\]

\[
\Rightarrow \frac{(y/x) - 1}{((y/x) + 3)^5} = Cx^4
\]

\[
\Rightarrow \frac{y - x}{(y + 3x)^5} = C.
\]

Therefore, the function \( y \) implicitly defined as \( C(y + 3x)^5 = y - x \), where \( C \) is a constant, is a solution to the given differential equation. Notice that in order to obtain the expression in (4), we implicitly assume that \( v \neq 1 \) and \( v \neq -3 \). These cases give rise to the functions \( y = x \) and \( y = -3x \), respectively. Thus, in order to obtain the complete set of solutions, we need to consider these two cases separately. It is then easy to check that they both are solutions to the given differential equation.

Hence, the solutions to the given differential equation are \( y = x, y = -3x, \) and \( C(y + 3x)^5 = y - x \). The direction field of this equation is given in Figure 4.

2.3.2. Let \( Q(t) \) denote the amount of salt (in grams) at the time \( t \). Then the rate of change for the salt in the tank is given by

\[
\frac{dQ}{dt} = R_{in} - R_{out}.
\]

Here, \( R_{in} = 2\gamma \text{ g/min} \) and \( R_{out} = \frac{2Q}{120} = \frac{Q}{60} \text{ g/min} \). Thus, we obtain the differential equation

\[
\frac{dQ}{dt} = 2\gamma - \frac{Q}{60} \Rightarrow \frac{dQ}{dt} + \frac{Q}{60} = 2\gamma.
\]

This is a first order linear differential equation with general solution

\[
Q(t) = 120\gamma + ce^{-t/60}.
\]

Under the initial condition \( Q(0) = 0 \), we obtain \( c = -120\gamma \). Thus, the solution to the IVP is

\[
Q(t) = 120\gamma \left( 1 - e^{-t/60} \right).
\]
Figure 4: The direction field of \( \frac{dy}{dx} = \frac{4y - 3x}{2x - y} \).

It is easy to see that \( \lim_{t \to \infty} Q(t) = 120 \gamma \).

2.3.10.

a. Let \( S(t) \) denote the amount of money the home buyer owe on his mortgage payment. Since the mortgage has a 6% annual interest rate, and the person can payoff at most \( 12 \times \$1,500 = \$18,000 \) a year, we obtain the differential equation for the growth rate, as follows.

\[
\frac{dS}{dt} = 0.06S - 18,000.
\]

The solution to this equation is

\[
S(t) = 300,000 + (S_0 - 300,000)e^{-0.06t}
\]

where \( S_0 = S(0) \) is the maximum amount the buyer can borrow.

If the mortgage is 20 years, then \( S(20) = 0 \) which gives \( S_0 \approx \$209,642 \). If the mortgage is 30 years, then \( S(30) = 0 \) which gives \( S_0 \approx \$250,410 \).

b. For the 20-years mortgage, the total interest is \( 20 \times 12 \times \$1500 - \$209,642 \approx \$150,358 \). Similarly, we can compute that, for the 30-years mortgage, the total interest is \( 20 \times 12 \times \$1500 - \$209,642 \approx \$289,590 \).

2.3.16. Let \( u(t) \) be the temperature of the cup at time \( t \) with \( u(0) = u_0 \), and let \( T \) be the room temperature. From the first assignment, we know that

\[
u(t) = T + (u_0 - T)e^{-kt}\]
for some rate $k$. 

Here, $u_0 = 200, T = 70,$ and $u(1) = 190$. So,

$$190 = 70 + (200 - 70)e^{-k} \Rightarrow k = \ln \left(\frac{13}{12}\right) \approx 0.0800427.$$ 

Let $\tau$ be the time needed for the cup to reach 150°F, then

$$150 = u(\tau) = 70 + 130e^{-0.0800427\tau} \Rightarrow \tau = \frac{\ln(13/8)}{0.0800427} \approx 6.06561 \text{ minutes}.$$ 

2.4.2. Consider the IVP $t(t - 4)y' + y = 0$ with $y(2) = 1$. We first transform the given equation into the “standard” form of first order linear differential equation, as follows.

$$t(t - 4)y' + y = 0 \Rightarrow y' + \frac{1}{t(t - 4)}y = 0$$

for $t \neq 0$ and $t \neq 4$.

We then use Theorem 2.4.1 (page 69) with $p(t) = \frac{1}{t(t - 4)}, g(t) = 0$, and $t_0 = 2$. Since $p(t)$ and $g(t)$ are both continuous on the open interval $(0, 4)$ that contains the point $t_0 = 2$, we know that the IVP is certain to have a solution on $0 < t < 4$.

2.4.13. Consider the IVP $y' = -\frac{4t}{y}$ with $y(0) = y_0$. For $y \neq 0$, we have

$$\frac{dy}{dt} = -\frac{4t}{y} \Rightarrow ydy = -4tdt$$

$$\Rightarrow \int ydy = \int -4t \, dt$$

$$\Rightarrow \frac{y^2}{2} = -2t^2 + c$$

$$\Rightarrow y(t) = \pm \sqrt{c - 4t^2}.$$ 

Under the initial condition, $y_0 = y(0) = \pm \sqrt{c} \Rightarrow c = y_0^2$. Thus, the solution to the IVP is

$$y(t) = \pm \sqrt{y_0^2 - 4t^2}.$$ 

Specifically, depending on the sign of $y_0$, we then have one of the following answers:

$$y(t) = \begin{cases} \sqrt{y_0^2 - 4t^2} & \text{if } y_0 > 0 \\ -\sqrt{y_0^2 - 4t^2} & \text{if } y_0 < 0. \end{cases}$$

In either case, the solution only exists when $y_0^2 > 4t^2 \Rightarrow |t| < \frac{|y_0|}{2}$, for $|y_0| \neq 0$.

2.5.3. The graph of $f(y) = y(y - 1)(y - 2)$ is given in Figure 5 below. The three equilibrium points are:
\[ y = 0 \text{ (unstable)}, \]
\[ y = 1 \text{ (stable), and} \]
\[ y = 2 \text{ (unstable)}. \]

Several solutions are given displayed in Figure 6.

Figure 5: The graph of \( f(y) = y(y - 1)(y - 2) \).

Figure 6: Solutions of \( \frac{dy}{dt} = y(y - 1)(y - 2) \).

2.5.6. The graph of \( f(y) = -\frac{2 \arctan y}{1 + y^2} \) is given in Figure 7. There is only one equilibrium point at \( y = 0 \). This is a stable equilibrium point. The graphs of several solution are given in Figure 8.

2.5.7. Consider the equation \( \frac{dy}{dt} = f(y) = k(1 - y)^2 \) where \( k > 0 \) is a constant.

a. It is easy to see that \( f(y) = 0 \iff y = 1 \). so \( y = 1 \) is the only critical point. Also observe that \( y \equiv 1 \) is a solution to the given differential equation, and thus, \( \phi(t) = 1 \) is the corresponding equilibrium solution to the critical point.
b. We have $y' = k(1 - y)^2 > 0$ for $k > 0$ and $y \neq 1$. It is then easy to see that $y$ is increasing as a function of $t$ for both $y > 1$ and $y < 1$. The graph of $f$ is given in Figure 9. Also, the direction field of this equation is given in Figure 10 below.

c. Using technique for solving separable equations,

$$\frac{dy}{dt} = k(1 - y)^2 \Rightarrow \frac{dy}{(1 - y)^2} = kdt$$

$$\Rightarrow \int \frac{dy}{(1 - y)^2} = \int kdt$$

$$\Rightarrow \frac{1}{1 - y} = kt + c$$

$$\Rightarrow y(t) = 1 - \frac{1}{kt + c}$$

For $t_0 = 0$, and the initial condition $y(0) = y_0$, we obtain $c = \frac{1}{1 - y_0}$. So, the solution to the IVP is

$$y(t) = \frac{kt(1 - y_0) + y_0}{kt(1 - y_0) + 1}.$$ 

This solution has a singularity at $t_s = \frac{1}{k(y_0 - 1)}$ and thus is only defined on either $(-\infty, t_s)$ or $(t_s, +\infty)$. The relative order of $t_0$ and $t_s$ will then decide which interval we choose.

If $y_0 < 1$ then $t_s = \frac{1}{k(y_0 - 1)} < 0$. So $t_s < t_0 < +\infty$. The solution to the given IVP is defined for each $t \in (t_s, +\infty)$. Therefore,

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} \frac{kt(1 - y_0) + y_0}{kt(1 - y_0) + 1} = 1.$$ 

If $y_0 > 1$ then $t_s = \frac{1}{k(y_0 - 1)} > 0$. So $-\infty < t_0 < t_s$. The solution to the given IVP is now defined for each $t \in (-\infty, t_s)$. Therefore,

$$\lim_{t \to t_s^-} y(t) = \lim_{t \to t_s^-} \frac{kt(1 - y_0) + y_0}{kt(1 - y_0) + 1} = +\infty.$$
Figure 8: The solutions of $\frac{dy}{dt} = -\frac{2\arctan y}{1+y^2}$.

Figure 9: The graph of $f(y) = k(1 - y)^2$ for several $k > 0$. 
Figure 10: Direction field for $\frac{dy}{dt} = (1 - y)^2$