2.5.10. The differential equation

\[ \frac{dy}{dt} = y(1 - y^2), \quad -\infty < y_0 < \infty \]

has three equilibrium solutions: \( y = 0 \) (unstable), \( y = -1 \) and \( y = 1 \) (stable). The direction field is given in Figure 1 below.

![Figure 1: The direction field of \( \frac{dy}{dt} = y(1 - y^2) \).](image)

2.5.15. We know that the logistic equation \( \frac{dy}{dt} = ry \left(1 - \frac{y}{K}\right) \) with \( y(0) = y_0 \) has solution

\[ y(t) = \frac{y_0 K}{y_0 + (K - y_0) e^{-rt}}. \]

a. If \( y_0 = K/3 \), let \( \tau \) be the time at which the initial population double, we then have

\[
\frac{2K}{3} = \frac{K^2/3}{K/3 + (2K/3)e^{-rt}} \Rightarrow \tau = \frac{1}{r} \ln\left(\frac{4}{0.025}\right) \approx 55.4518 \text{ (years)}.
\]

b. Let \( y_0 / K = \alpha \) and suppose \( y(T) / K = \beta \) for \( 0 < \alpha, \beta < 1 \). Then,

\[
\beta K = y(T) = \frac{\alpha K^2}{\alpha K + (K - \alpha K) e^{-rT}} \Rightarrow T = \frac{1}{r} \ln \left[ \frac{\beta (1 - \alpha)}{\alpha (1 - \beta)} \right].
\]

When \( \alpha = 0.1, \beta = 0.9, \) and \( r = 0.025 \),

\[ T = \frac{1}{r} \ln \left[ \frac{\beta (1 - \alpha)}{\alpha (1 - \beta)} \right] \approx 175.778 \text{ (years)}. \]
2.5.20. Consider the Schaefer model given by
\[
\frac{dy}{dt} = f(y) = ry \left( 1 - \frac{y}{K} \right) - Ey, \quad \text{for} \ r, E > 0.
\]

\[\text{a.} \quad \text{The equilibrium solution of this equation is given by}
\]
\[
y_1 = 0 \quad \text{or} \quad y_2 = K \left( 1 - \frac{E}{r} \right).
\]

If \( E < r \) then it is easy to see that \( y_2 > 0 \).

\[\text{b. For the equilibrium} \ y_1 = 0, \ \text{we have} \ f(y) < 0 \ \text{on the left of 0 and} \ f(y) > 0 \ \text{on the right of 0 so} \ y_1 = 0 \ \text{is an unstable equilibrium. For the equilibrium} \ y_2 = K \left( 1 - \frac{E}{r} \right) = y_s, \ \text{we have} \ f(y) > 0 \ \text{on the left of} \ y_s \ \text{and} \ f(y) < 0 \ \text{on the right of} \ y_s \ \text{so} \ y_2 = K \left( 1 - \frac{E}{r} \right) \ \text{is an asymptotically stable equilibrium.}
\]

c. \( Y = Ey_2 = EK \left( 1 - \frac{E}{r} \right) = -\frac{KE^2}{r} + EK. \)

d. Through simple manipulation of the terms in \( Y, \)
\[
Y = -\frac{KE^2}{r} + EK = -\frac{K}{r} \left( E^2 - rE \right) = -\frac{K}{r} \left( E^2 - 2 \frac{r}{2} E + \frac{r^2}{4} \right) + \frac{Kr}{4} = -\frac{K}{r} \left( E - \frac{r}{2} \right)^2 + \frac{Kr}{4} \leq \frac{Kr}{4}.
\]

So the maximum sustainable yield \( Y_m = \frac{Kr}{4} \) when \( E = \frac{r}{2}. \)

2.6.1. Consider the equation
\[
(2x + 3) + (2y - 2)y' = M(x, y) + N(x, y)y' = 0.
\]

We have \( M_y(x, y) = 0 = N_x(x, y) \) so this is an exact equation. We want to find a function \( \psi(x, y) \)
such that \( \psi_x(x, y) = M(x, y) \) and \( \psi_y(x, y) = N(x, y). \) So,
\[
\psi(x, y) = \int M(x, y)dx = \int (2x + 3)dx = x^2 + 3x + h(y).
\]

Thus,
\[
2y - 2 = N(x, y) = \psi_y(x, y) = h'(y) \Rightarrow h(y) = \int (2y - 2)dy = y^2 - 2y + c.
\]

Therefore,
\[
\psi(x, y) = x^2 + 3x + y^2 - 2y + c.
\]

The given exact equation then has solutions given implicitly by
\[
x^2 + 3x + y^2 - 2y = C.
\]
for any constant $C$.

**2.6.8.** Consider the equation

$$(e^x \sin y + 3y) - (3x - e^x \sin y)y' = M(x, y) + N(x, y)y' = 0.$$  

We have $M_y(x, y) = e^x \cos y + 3 \neq -3 + e^x \sin y = N_x(x, y)$ so this is *not* an exact equation.

**2.6.10.** Let $x > 0$, consider the equation

$$\left(\frac{y}{x} + 6x\right) - (\ln x - 2)y' = M(x, y) + N(x, y)y' = 0.$$  

We have $M_y(x, y) = 1/x = N_x(x, y)$ so this is an exact equation. So,

$$\psi(x, y) = \int M(x, y)dx = \int \left(\frac{y}{x} + 6x\right) dx = y \ln x + 3x^2 + h(y).$$  

Thus,

$$\ln x - 2 = N(x, y) = \psi_y(x, y) = \ln x + h'(y) \Rightarrow h'(y) = -2 \Rightarrow h(y) = \int -2dy = 2y + c.$$  

Therefore, the given exact equation then has solutions given implicitly by

$$y \ln x + 3x^2 - 2y = C.$$  

**2.6.13.** Consider the equation

$$(2x - y) + (2y - x)y' = M(x, y) + N(x, y)y' = 0.$$  

We have $M_y(x, y) = -1 = N_x(x, y)$ so this is an exact equation. So,

$$\psi(x, y) = \int M(x, y)dx = \int (2x - y)dx = x^2 + xy + h(y).$$  

Thus,

$$2y - x = N(x, y) = \psi_y(x, y) = x + h'(y) \Rightarrow h'(y) = 2y - 2x \Rightarrow h(y) = \int (2y - 2x)dy = y^2 - 2xy + c.$$  

Therefore, the given exact equation then has solutions given implicitly by $x^2 + y^2 - xy = C$. Under the initial condition $y(1) = 3$, we obtain $C = 7$. Thus, the solution to the IVP (implicitly) is $y^2 - xy + x^2 - 7 = 0$.

Writing $y$ in terms of $x$ yields $y = \frac{x \pm \sqrt{28 - 3x^2}}{2}$. Under the initial conditions $x_0 = 1$ and $y_0 = 3$, only the solution with the positive sign remains and this solution is defined for $-\sqrt{28/3} \leq x \leq \sqrt{28/3}$. However, one can check that when $x = \pm \sqrt{28/3}$, we obtain $y = x/2$ which is *not* a solution to the given equation.
Hence, the solution to the IVP is
\[ y(t) = \frac{x + \sqrt{28 - 3x^2}}{2}, \]
defined for \( x \in (-\sqrt{28/3}, \sqrt{28/3}) \).

2.6.15. Consider the equation
\[
(xy^2 + bx^2) + (x^3 + x^2y)y' = M(x, y) + N(x, y)y' = 0.
\]
We have \( M_y(x, y) = 2xy + bx^2 \) and \( N_x(x, y) = 3x^2 + 2xy \). This is an exact equation if and only if \( b = 3 \). So, when \( b = 3 \),
\[
\psi(x, y) = \int M(x, y)dx = \int (xy^2 + 3x^2y)dx = \frac{1}{2} x^2y^2 + x^3y + h(y).
\]
Thus,
\[
x^3 + x^2y = \psi_y(x, y) = x^3 + x^2y + h'(y) \Rightarrow h'(y) = 0 \Rightarrow h(y) = c.
\]
Therefore, the solution to the exact equation is \( x^2y^2 + 2x^3y = C \).

2.6.30. Consider the equation
\[
\left( \frac{4x^3}{y^2} + \frac{3}{y} \right) + \left( \frac{3x}{y^2} + 4y \right) y' = M(x, y) + N(x, y)y' = 0.
\]
First observe that
\[
\frac{N_x(x, y) - M_y(x, y)}{M(x, y)} = \frac{3}{y^2} - \left( \frac{-8x^3}{y^3} - \frac{3}{y^2} \right) = \frac{2}{y'}
\]
so the integrating factor \( \mu \) depends only on \( y \). This integrating factor is then given by
\[
\mu(y) = \exp \left( \int \frac{N_x(x, y) - M_y(x, y)}{M(x, y)} dy \right) = \exp \left( \int \frac{2}{y} dy \right) = e^{\ln(y^2)} = y^2.
\]
Under the integrating factor \( \mu(y) = y^2 \), the given equation becomes
\[
(4x^3 + 3y) + (3x + 4y^3)y' = 0. \tag{1}
\]
It is easy to check that (1) is an exact equation. So,
\[
\psi(x, y) = \int (4x^3 + 3y)dx = x^4 + 3xy + h(y).
\]
Thus,
\[
3x + 4y^3 = \psi_y(x, y) = 3x + h'(y) \Rightarrow h'(y) = 4y^3 \Rightarrow h(y) = y^4 + c.
\]
Hence, the solution is \( x^4 + 3xy + y^4 = C \).

3.1.1. Consider the second order linear equation
\[ y'' + 2y' - 3y = 0. \]
The characteristic equation is given by
\[ r^2 + 2r - 3r = 0 \]
which gives solutions \( r_1 = -1 \) and \( r_2 = 3 \). Hence, the general solution of the given equation is
\[ y(t) = c_1 e^{-t} + c_2 e^{3t}. \]

3.1.11. Consider the second order linear equation
\[ 6y'' - 5y' + y = 0, \quad y(0) = 4, \quad y'(0) = 0. \]
The characteristic equation is given by
\[ 6r^2 - 5r + r = 0 \]
which gives solutions \( r_1 = 1/3 \) and \( r_2 = 1/2 \). Hence, the general solution is
\[ y(t) = c_1 e^{t/3} + c_2 e^{t/2}. \]

Under the initial conditions, we have
\[
\begin{align*}
4 &= y(0) = c_1 + c_2 \\
0 &= y'(0) = \frac{c_1}{3} + \frac{c_2}{2}
\end{align*}
\]
which gives \( c_1 = 12 \) and \( c_2 = -8 \). Hence, the solution to the IVP is
\[ y(t) = 12e^{t/3} - 8e^{t/2}. \]
The graph of this solution is given in Figure 2. It is easy to see that when \( t \to \infty \), the long term behavior of the solution is controlled by the \( e^{t/2} \) term. Thus, \( y(t) \to -\infty \) as \( t \to \infty \).

3.1.18. Let \( y \) be a function of \( t \) and suppose that the second order linear differential equation \( y'' + ay' + by = 0 \) has general solution given by \( y(t) = c_1 e^{-t/2} + c_2 e^{-2t} \).
The characteristic equation for this second order equation is \( r^2 + ar + b = 0 \). In addition, the powers of the exponentials in the solution tell us that \( r_1 = -1/2 \) and \( r_2 = -2 \) must be solutions to this characteristic equation. This gives

\[
-a = r_1 + r_2 = \frac{-1}{2} - 2 = -\frac{5}{2} \quad \text{and} \quad b = r_1r_2 = \left( -\frac{1}{2} \right) (-2) = 1.
\]

Thus, the required differential equation is

\[
y'' + \frac{5}{2} y' + y = 0 \Rightarrow 2y'' + 5y' + 2y = 0.
\]

3.1.21. Consider the second order linear equation

\[
y'' - y' - 2y = 0, \quad y(0) = \alpha, \quad y'(0) = 2.
\]

The characteristic equation is given by

\[
r^2 - r - 2r = 0
\]

which gives solutions \( r_1 = 2 \) and \( r_2 = -1 \). Hence, the general solution is

\[
y(t) = c_1e^{2t} + c_2e^{-t}.
\]

Under the initial conditions, we have

\[
\begin{align*}
\alpha &= y(0) = c_1 + c_2 \\
2 &= y'(0) = 2c_1 - c_2
\end{align*}
\]

which gives \( c_1 = \frac{2 + \alpha}{3} \) and \( c_2 = \frac{2\alpha - 2}{3} \). Therefore, the solution of the given IVP, in terms of \( \alpha \) is

\[
y(t) = \frac{2 + \alpha}{3} e^{2t} + \frac{2\alpha - 2}{3} e^{-t}.
\] \tag{2}

Observe that if the coefficient of \( e^{2t} \) is nonzero then the long term behavior of the solution in (2) will be controlled by this term, which goes to \(+\infty\) as \( t \to \infty \). Therefore, in order for the solution in (2) to approach zero as \( t \to \infty \), we must have \((\alpha + 2)/3 = 0\) which yields \( \alpha = -2 \).

3.2.3. The Wronskian of \( y_1(t) = e^{-2t} \) and \( y_2(t) = te^{-2t} \) is given by

\[
W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = (e^{-2t})(e^{-2t} - 2te^{-2t}) - (te^{-2t})(-2e^{-2t}) = e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t}.
\]

3.2.5. The Wronskian of \( y_1(t) = e^t \sin t \) and \( y_2(t) = e^t \cos t \) is given by

\[
W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t \sin t + e^t \cos t & e^t \cos t - e^t \sin t \end{vmatrix} = (e^t \sin t)(e^t \cos t - e^t \sin t) - (e^t \cos t)(e^t \cos t + e^t \sin t) = e^{2t} \sin t \cos t - e^{2t} \sin^2 t - e^{2t} \cos^2 t - e^{2t} \cos t \sin t = -e^{2t}(\sin^2 t + \cos^2 t) = -e^{2t}.
\]
3.2.9. Consider the IVP

\[ t(t - 4)y'' + 3ty' + 4y = 2, \quad y(3) = 0, \quad y'(3) = -1. \]

This given differential equation can be written as

\[ y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = y'' + p(t)y' + q(t)y = 0 \]

Observe that \( p(t) \) and \( q(t) \) are continuous on the open interval \((0, 4)\) that contains the initial value \( t_0 = 3 \). So by Theorem 3.2.1 (page 146), the given differential equation has a unique solution for \( 0 < t < 4 \).

3.2.13. For \( t > 0 \), consider the differential equation \( t^2y'' - 2y = 0 \) and two solutions \( y_1(t) = t^2 \) and \( y_2(t) = t^{-1} \). We have,

- \( y_1' = 2t, y_1'' = 2 \Rightarrow t^2y_1'' - 2y_1 = 2t^2 - 2t^2 = 0 \). Thus, \( t^2 \) is a solution of the given equation.

- \( y_2' = -t^{-2}, y_2'' = 2t^{-3} \Rightarrow t^2y_2'' - 2y_2 = 2t^2t^{-3} - 2t^{-1} = 2t^{-1} - 2t^{-1} = 0 \). Thus, \( t^{-1} \) is also a solution of the given equation.

By Theorem 3.2.2 (page 147), if \( y_1 \) and \( y_2 \) are two independent solutions to a second order linear differential equation then any linear combination \( c_1 y_1 + c_2 y_2 \) is also a solution to the same equation. Hence, for any constants \( c_1 \) and \( c_2 \), \( y(t) = c_1 t^2 + c_2 t^{-1} \) is also a solution to the given differential equation.