3.2.25. Consider the differential equation \( y'' - 2y' + y = 0 \).
For \( y_1(t) = e^t \), we have \( y'_1(t) = y''(t) = e^t \) so
\[
y'_1 - 2y'_1 + y_1 = e^t - 2e^t + e^t = 0.
\]
Thus, \( y_1(t) = e^t \) is a solution.
For \( y_2(t) = te^t \), we have \( y'_2(t) = te^t + e^t \) and \( y''_2(t) = 2e^t + te^t \) so
\[
y''_2 - 2y'_2 + y_2 = (2e^t + te^t) - 2(e^t + te^t) + te^t = 0.
\]
Thus, \( y_2(t) = e^t \) is also a solution.
The Wronskian is given by
\[
W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = (e^t)(e^t + te^t) - (te^t)(e^t) = e^{2t} \neq 0.
\]
\( W \neq 0 \) shows that \( y_1 \) and \( y_2 \) form a fundamental set of solutions.

3.2.27. Consider the differential equation \( (1 - x \cot x)y'' - xy' + y = 0 \) for \( 0 < x < \pi \).
For \( y_1(x) = x \), we have \( y'_1(x) = 1 \) and \( y''_1(x) = 0 \) so
\[
(1 - x \cot x)y''_1 - xy'_1 + y_1 = 0 - x(1) + x = 0.
\]
Thus, \( y_1(t) = x \) is a solution.
For \( y_2(x) = \sin x \), we have \( y'_2(x) = \cos x \) and \( y''_2(x) = -\sin x \) so
\[
(1 - x \cot x)y''_2 - xy'_2 + y_2 = (1 - x \cot x)(-\sin x) - x \cos x + \sin x
\]
\[=- \sin x + x \cos x - x \cos x + \sin x = 0.
\]
Thus, \( y_2(t) = \sin x \) is also a solution.
The Wronskian is given by
\[
W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = (x)(\cos x) - (1)(\sin x) \neq 0.
\]
\( W \neq 0 \) shows that \( y_1 \) and \( y_2 \) form a fundamental set of solutions.

3.3.4. \( e^{2-(\pi/2)i} = e^2 \cos \left( \frac{\pi}{2} \right) - i \sin \left( \frac{\pi}{2} \right) = -e^2i. \)

3.3.8. Consider the differential equation \( y'' - 2y' + 6y = 0 \). The characteristic equation is given by \( r^2 - 2r + 6 = 0 \) which has complex roots \( r = 1 \pm i\sqrt{5} \). Hence, the general solution is
\[
y(t) = c_1e^t \cos(\sqrt{5}t) + c_2e^t \sin(\sqrt{5}t).
\]

3.3.19. Consider the differential equation \( y'' - 2y' + 5y = 0 \). The characteristic equation is given by \( r^2 - 2r + 5 = 0 \) which has complex roots \( r = 1 \pm 2i \). Hence, the general solution is
\[
y(t) = c_1e^t \cos(2t) + c_2e^t \sin(2t).
\]
Thus, the characteristic equation

$\lambda^2 - \alpha \lambda + \beta = 0$.

So $y(t) = c_2 e^{\lambda t} \sin(2t)$ and thus, $y(t) = 2c_2 e^{\lambda t} \sin(2t) + 2c_2 e^{\lambda t} \cos(2t)$. Under the initial condition $y'(\pi/2) = 2$, we have

$$2 = y(\pi/2) = -2c_2 e^{\lambda t/2} \Rightarrow c_2 = -e^{-\pi/2}.$$ Hence, the solution to the IVP is $y(t) = -e^{t - \pi/2} \sin(2t)$. This solution is given in Figure 1. This is a growing oscillation as $t \to \infty$.

![Figure 1: The graph of $y(t) = -e^{t - \pi/2} \sin(2t)$.

3.3.32. Consider the differential equation $ay'' + by' + cy = 0$ with $b^2 - 4ac < 0$. Further suppose that the characteristic equation $ar^2 + br + c = 0$ has complex roots $\lambda \pm i\mu$ so that

$$a(\lambda \pm i\mu)^2 + b(\lambda \pm i\mu) + c = 0.$$ For $u(t) = e^{\lambda t} \cos(\mu t) = e^{\lambda t} \left( \frac{e^{i\mu t} + e^{-i\mu t}}{2} \right) = \frac{1}{2} \left( e^{t(\lambda + i\mu)} + e^{t(\lambda - i\mu)} \right)$, we have

$$u'(t) = \frac{1}{2} \left( (\lambda + i\mu)e^{t(\lambda + i\mu)} + (\lambda - i\mu)e^{t(\lambda - i\mu)} \right) \text{ and}$$

$$u''(t) = \frac{1}{2} \left( (\lambda + i\mu)^2 e^{t(\lambda + i\mu)} + (\lambda - i\mu)^2 e^{t(\lambda - i\mu)} \right)$$

Thus,

$$ay'' + by' + cy = \frac{a}{2} \left( (\lambda + i\mu)^2 e^{t(\lambda + i\mu)} + (\lambda - i\mu)^2 e^{t(\lambda - i\mu)} \right)$$

$$+ \frac{b}{2} \left( (\lambda + i\mu)e^{t(\lambda + i\mu)} + (\lambda - i\mu)e^{t(\lambda - i\mu)} \right) + \frac{c}{2} \left( e^{t(\lambda + i\mu)} + e^{t(\lambda - i\mu)} \right)$$

$$= \frac{1}{2} e^{t(\lambda + i\mu)} \left( a(\lambda + i\mu)^2 + b(\lambda + i\mu) + c \right)$$

$$+ \frac{1}{2} e^{t(\lambda - i\mu)} \left( a(\lambda - i\mu)^2 + b(\lambda - i\mu) + c \right) = 0.$$ So $u(t) = e^{\lambda t} \cos(\mu t)$ is a solution.

Similar argument can be applied to show that

$$u(t) = e^{\lambda t} \sin(\mu t) = e^{\lambda t} \left( \frac{e^{i\mu t} - e^{-i\mu t}}{2i} \right) = \frac{1}{2i} \left( e^{t(\lambda + i\mu)} - e^{t(\lambda - i\mu)} \right).
is also a solution to the given differential equation.

3.3.34. Consider the equation

\[ t^2 \frac{d^2 y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = 0 \]  \hspace{1cm} (1)

for real constants \( \alpha \) and \( \beta \).

a. Let \( x = \ln(t) \) then

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{1}{t} \frac{dy}{dx} \quad \text{and} \quad \frac{d^2 y}{dt^2} = \frac{d}{dx} \left( \frac{dy}{dt} \right)^2 \frac{d^2 y}{dx^2} = -\frac{1}{t^2} \frac{dy}{dx} + \frac{1}{t^2} \frac{d^2 y}{dx^2}.
\]

b. From part (a.), we have \( t \frac{dy}{dt} = \frac{dy}{dx} \) and \( t^2 \frac{d^2 y}{dt^2} = \frac{d^2 y}{dx^2} - \frac{dy}{dx} \). Therefore, (1) becomes

\[ 0 = t^2 \frac{d^2 y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = \frac{d^2 y}{dx^2} - \frac{dy}{dx} + \alpha \frac{dy}{dx} + \beta y = \frac{d^2 y}{dx^2} + (\alpha - 1) \frac{dy}{dx} + \beta y. \]

3.3.38. For \( t > 0 \), consider the differential equation

\[ t^2 y'' - 4ty' - 6y = 0. \]  \hspace{1cm} (2)

Let \( x = \ln(t) \) then we can transform the given equation (2) into

\[ \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} - 6y = 0. \]  \hspace{1cm} (3)

The characteristic equation of (3) is \( r^2 - 5r - 6 = 0 \) which has two distinct real solutions \( r_1 = -1 \) and \( r_2 = 6 \). Thus, the general solution of (3) is

\[ y(x) = c_1 e^{-x} + c_2 e^{6x}. \]

Hence, the general solution of (2) is given by

\[ y(t) = c_1 e^{-\ln(t)} + c_2 e^{6\ln(t)} = c_1 t^{-1} + c_2 t^6. \]

3.4.2. Consider the differential equation \( 9y'' + 6y' + y = 0 \). The characteristic equation is \( 9r^2 + 6r + 1 = 0 \) which has repeated root \( r = -\frac{1}{3} \). Thus, the general solution is given by

\[ y(t) = c_1 e^{-\frac{t}{3}} + c_2 te^{-\frac{t}{3}}. \]

3.4.12. Consider the differential equation \( y'' - 6y' + 9y = 0 \). The characteristic equation is \( r^2 - 6r + 9 = 0 \) which has repeated root \( r = 3 \). Thus, the general solution is given by

\[ y(t) = c_1 e^{3t} + c_2 te^{3t}. \]
Figure 2: The graph of $y(t) = 2te^{3t}$.

Under the initial condition $y(0) = 0$, we have $c_1 = 0$.
Thus, $y(t) = c_2te^{3t}$ which gives $y'(t) = c_2e^{3t} + 3c_2te^{3t}$. Under the initial condition $y'(0) = 2$, we have $c_2 = 2$.
Hence, the solution to the IVP is $y(t) = 2te^{3t}$. The graph of this solution is given in Figure 2. As $t \to \infty$, we have $y(t) \to \infty$.

3.4.16. Consider the differential equation $y'' - y' + 0.25y = 0$. The characteristic equation is $r^2 - r + \frac{1}{4} = 0$ which has repeated root $r = 1/2$. Thus, the general solution is given by

$$y(t) = c_1e^{t/2} + c_2te^{t/2}.$$

Under the initial condition $y(0) = 2$, we have $c_1 = 2$.
Thus, $y(t) = 2e^{t/2} + c_2te^{t/2}$ which gives $y'(t) = \frac{e^{t/2}}{2} + c_2e^{t/2} + \frac{c_2}{2} te^{t/2}$. Under the initial condition $y'(0) = b$, we have $b = 1 + c_2 \Rightarrow c_2 = b - 1$.
Hence, the solution to the IVP is

$$y(t) = 2e^{t/2} + (b - 1)te^{t/2}.$$

Observe that when $b \neq 1$, the long-term behavior of $y$ is controlled by the term $(b - 1)te^{t/2}$. So if $b < 1$ then $y \to -\infty$ as $t \to \infty$; whereas if $b \geq 1$ then $y \to \infty$ as $t \to \infty$. This shows that $b = 1$ is the critical value.