1.3.12. To determine if \( \mathbf{b} \) is a linear combination of \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \), we need to find \( x_1, x_2, x_3 \) such that:

\[
x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3 = \mathbf{b}.
\]

From the definition of scalar multiplication and vector addition, we have

\[
\begin{pmatrix}
1 \\
2 \\
2
\end{pmatrix} +
\begin{pmatrix}
0 \\
5 \\
8
\end{pmatrix} +
\begin{pmatrix}
2 \\
0 \\
8
\end{pmatrix} =
\begin{pmatrix}
-5 \\
11 \\
-7
\end{pmatrix}
\]

The values \( x_1, x_2, x_3 \) make the vector equation (1) true if and only if \( x_1, x_2, x_3 \) satisfy the system

\[
\begin{align*}
x_1 + 2x_3 &= -5 \\
-2x_1 + 5x_2 &= 11 \\
2x_1 + 5x_2 + 8x_3 &= -7
\end{align*}
\]

To solve (2), row reduce the augmented matrix of the system:

\[
\begin{pmatrix}
1 & 0 & 2 & -5 \\
-2 & 5 & 0 & 11 \\
2 & 5 & 8 & -7
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 0 & 2 & -5 \\
0 & 5 & 4 & 1 \\
0 & 5 & 4 & 3
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 0 & 2 & -5 \\
0 & 5 & 4 & 1 \\
0 & 0 & 0 & 2
\end{pmatrix}
\]

Since the rightmost column of the augmented matrix is a pivot column, by the Existence and Uniqueness Theorem, (2) is inconsistent.

Hence, \( \mathbf{b} \) is not a linear combination of \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \).

1.3.18. Again, we need to find \( x_1 \) and \( x_2 \) such that:

\[
x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 = \mathbf{y}.
\]

Following the same steps as problem 1.3.12, we obtain the following augmented matrix.

\[
\begin{pmatrix}
1 & -3 & h \\
0 & 1 & -5 \\
-2 & 8 & -3
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & -3 & h \\
0 & 1 & -5 \\
0 & 2 & 2h - 3
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & -3 & h \\
0 & 1 & -5 \\
0 & 0 & 2h + 7
\end{pmatrix}
\]

To avoid a pivot on the rightmost column of the augmented matrix, we require \( 2h + 7 = 0 \), which gives \( h = -7/2 \). Thus, when \( h = -7/2 \) then \( \mathbf{y} \) is a linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \).
1.3.22. There are many. Here is one of such examples:

\[ A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \]

1.3.23.

a. False: \([-4 \ 3] \neq [-4 \ 3]\).

b. False: Just plot the vectors to see this.

c. True: \(\frac{1}{2} \cdot v_1 = \frac{1}{2} \cdot v_1 + 0 \cdot v_2\).

d. True: Look at the top box of page 30.

e. False: The statement fails when \(u\) and \(v\) are linearly dependent.

1.3.24.

a. True: Look at the definition of vectors in \(\mathbb{R}^n\) and take \(n = 5\).

b. True: \(v + (u - v) = u\).

c. False: Review the definition of linear combinations. There is nothing that prevents us from having all weights \(c_1 = c_2 = \ldots = c_p = 0\). In such case, we simply have the zero vector as a linear combination of the vectors \(v_1, \ldots, v_p\).

d. True: Look at Figure 11 (page 30) and the comment above it.

e. True: Look at the definition of Span on page 30.

1.3.26.

a. To see if \(b\) is in \(W\), we need to check whether \(b\) is a linear combination of the columns of \(A\). That is, we need to find \(x_1, x_2, x_3\) such that:

\[ x_1 a_1 + x_2 a_2 + x_3 a_3 = b. \]

Following the same steps as in 1.3.12, we obtain:

\[ 2 \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} + 1 \cdot \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix} \]

Hence, \(b \in W\).
b. It is easy to see that $a_3 = 0 \cdot a_1 + 0 \cdot a_2 + 1 \cdot a_3$. Thus, $a_3$ is a linear combination of the columns of $A$ under the weights $c_1 = c_2 = 0$ and $c_3 = 1$. Hence, $a_2 \in W$.

1.4.1. The product is **undefined** because the dimensions do not match.

1.4.10. Rewrite the system

\[
\begin{align*}
8x_1 - x_2 &= 4 \\
5x_1 + 4x_2 &= 1 \\
x_1 - 3x_2 &= 2
\end{align*}
\]

As a vector equation: $x_1 \cdot \begin{bmatrix} 8 \\ 5 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

As a matrix equation: $\begin{bmatrix} 8 & -1 \\ 5 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

1.4.12. The augmented matrix is:

\[
\begin{bmatrix}
1 & 2 & 1 & 0 \\
-3 & -1 & 2 & 1 \\
0 & 5 & 3 & -1
\end{bmatrix}
\]

From the Row Reduction Algorithm, we have:

\[
\begin{bmatrix}
1 & 2 & 1 & 0 \\
-3 & -1 & 2 & 1 \\
0 & 5 & 3 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 1 & 0 \\
0 & 5 & 5 & 1 \\
0 & 0 & -2 & -2
\end{bmatrix}
\]

Solving the system using back-substitution, we get: $x_3 = 1, x_2 = -4/5$, and $x_1 = 3/5$.

Hence, the solution, written as a vector, is:

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix}
\]

1.4.13. To determine if $u$ is in the plane in $\mathbb{R}^3$ spanned by the columns of $A$, we need to check whether $u$ is a linear combination of the columns of $A$. That is equivalent to checking whether the equation $Ax = u$ has a solution.

The augmented matrix is:

\[
\begin{bmatrix}
3 & -5 & 0 \\
-2 & 6 & 4 \\
1 & 1 & 4
\end{bmatrix}
\]
From the Row Reduction Algorithm, we have:

\[
\begin{bmatrix}
3 & -5 & 0 \\
-2 & 6 & 4 \\
1 & 1 & 4
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 4 \\
0 & 2 & 3 \\
0 & 0 & 0
\end{bmatrix}
\]

Since the rightmost column of the augmented matrix is not a pivot column, by the Existence and Uniqueness Theorem, the equation \(Ax = u\) is consistent.

Hence, \(u\) is a linear combination of the columns of \(A\) and thus is in the plane in \(\mathbb{R}^3\) spanned by the columns of \(A\).

1.4.18. First we use the row reduction algorithm to transform \(B\) into the row echelon form:

\[
\begin{bmatrix}
1 & 3 & -2 & 2 \\
0 & 1 & 1 & -5 \\
1 & 2 & -3 & 7 \\
-2 & -8 & 2 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 3 & -2 & 2 \\
0 & 1 & 1 & -5 \\
0 & -1 & -1 & 5 \\
0 & -2 & -2 & 3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 3 & -2 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 7 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Since \(B\) doesn’t have a pivot position in every row (it only has 3 pivots - highlighted above in boldface numbers), by Theorem 4, the columns of \(A\) do not span \(\mathbb{R}^4\). Also, the equation \(Bx = y\) does not have a solution for each \(y \in \mathbb{R}^4\).

1.4.24.

a. True: This is Theorem 3, first part.

b. True: Look at the definition of \(Ax\) on page 35.

c. True: Theorem 3, second part.

d. True: Simply negate all statements of Theorem 4. The equation \(Ax = b\) is inconsistent \(\iff\) \(b\) is not a linear combination of the columns of \(A\) \(\iff\) \(b\) is not in the set spanned by the columns of \(A\).

e. False: Look at the warning in the second paragraph of page 38.

f. True: By Theorem 4, if \(A\) is an \(m \times n\) matrix whose columns do not span \(\mathbb{R}^m\), then the equation \(Ax = b\) is inconsistent for some \(b\).

\footnote{In Math 109, this is called contrapositive.}
1.4.26. We have:

\[ 3u - 5v - w = 0 \Rightarrow 3u - 5v = w \]

\[ \Rightarrow 3 \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix} - 5 \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \]

Hence \( x_1 = 2 \) and \( x_2 = -3 \).

1.4.32. A set of three vectors in \( \mathbb{R}^4 \) can not span \( \mathbb{R}^4 \).

Proof: By contradiction, suppose there exists a set of three vectors \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \) in \( \mathbb{R}^4 \) such that \( \{ \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \} \) spans \( \mathbb{R}^4 \). Now let \( \mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3] \) and notice that \( \mathbf{A} \) is a \( 4 \times 3 \) matrix (with 3 columns and 4 rows).

By Theorem 4, in order for \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \), the matrix \( \mathbf{A} \) must have a pivot position in every row. This implies that \( \mathbf{A} \) needs at least four columns, contradicting with \( \mathbf{A} \) being a \( 4 \times 3 \) matrix. \( \square \)

In general, a set of \( n \) vectors in \( \mathbb{R}^m \) can not span \( \mathbb{R}^m \) when \( n < m \).

1.5.6. First we use the row reduction algorithm to transform \( [A \ 0] \) into the row echelon form:

\[ [A \ 0] = \begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

We can see that the pivot columns are 1 and 2. Therefore, \( x_1 \) and \( x_2 \) are basic variables while \( x_3 \) is free. By solving for the basic variables, we can easily obtain the general solution as followed:

\[ \begin{cases} x_1 = -4x_3 \\ x_2 = 3x_3 \\ x_3 \text{ is free} \end{cases} \]

The solution set in parametric vector form is:

\[ \mathbf{x} = x_3 \cdot \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} \]

1.5.11. From the row echelon form of \( \mathbf{A} \), we know that \( x_1, x_3, x_5 \) are basic variable while \( x_2, x_4, x_6 \) are free.
By solving for the basic variables, we can easily obtain the general solution as followed:

\[
\begin{align*}
    x_1 &= 4x_2 - 5x_6 \\
    x_2 &\text{ is free} \\
    x_3 &= x_6 \\
    x_4 &\text{ is free} \\
    x_5 &= 4x_6 \\
    x_6 &\text{ is free}
\end{align*}
\]

The solution set in parametric vector form is:

\[
x = x_2 \cdot \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \cdot \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}
\]

1.5.16. First we use the row reduction algorithm to transform \([A \ b]\) into the row echelon form:

\[
[A \ 0] = \begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

We can see that the pivot columns are 1 and 2. Therefore, \(x_1\) and \(x_2\) are basic variables while \(x_3\) is free. By solving for the basic variables, we can easily obtain the general solution as followed:

\[
\begin{align*}
    x_1 &= -5 - 4x_3 \\
    x_2 &= 3 + 3x_3 \\
    x_3 &\text{ is free}
\end{align*}
\]

The solution set in parametric vector form is:

\[
x = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}
\]

Hence, in \(\mathbb{R}^3\), the solution set is the line through \(\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}\) and parallel to the solution of the homogeneous system from Exercise 1.5.6.

1.5.18. For the homogeneous equation \(x_1 - 3x_2 + 5x_3 = 0\), the basic variable is \(x_1\) while \(x_2\) and \(x_3\) are free. As a vector, the general solution is:

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}
\]
Similarly, for the non-homogeneous equation \( x_1 - 3x_2 + 5x_3 = 4 \), the general solution in parametric vector form is:

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 + 3x_2 - 5x_3 \\ 0 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}
\]

Hence, in \( \mathbb{R}^3 \), the solution of the non-homogeneous equation is the plane through \( \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \) and parallel to the solution of the homogeneous equation.

1.5.24.

a. False: The nontrivial solution only requires \textit{at least one} entry to be nonzero.

b. True: This is similar to Example 2 on page 44.

c. True: If the zero vector is a solution then \( \mathbf{b} = A \cdot \mathbf{x} = A \cdot \mathbf{0} = \mathbf{0} \).

d. True: Look at the comment on page 46, after Example 3.

e. False: This fails if \( A \mathbf{x} = \mathbf{b} \) is inconsistent. Notice that while the equation \( A \mathbf{x} = \mathbf{0} \) is always consistent (why?), it may or may not be the same for \( A \mathbf{x} = \mathbf{b} \).

1.5.29. Let \( A \) be a \( 3 \times 3 \) matrix with three pivot positions.

a. No. When \( A \) is a \( 3 \times 3 \) matrix with three pivot positions, there must be a pivot on each column. Thus, all three columns of \( A \) are pivot column and there is no free variable for the equation \( A \mathbf{x} = \mathbf{0} \). So the trivial solution is the only solution.

b. Yes. In this case, there is a pivot position in every row. Thus, by Theorem 1.4 on page 37, the equation \( a \mathbf{x} = \mathbf{b} \) is consistent for every \( \mathbf{b} \).

1.5.30. Let \( A \) be a \( 3 \times 3 \) matrix with two pivot positions.

a. Yes. With two pivot positions and three columns, \( A \) must have a non-pivot column. Thus, one of the three variables of the equation \( A \mathbf{x} = \mathbf{0} \) is free. This leads to non-trivial solutions.

b. No. Here’s a counter example. Consider \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) and \( \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \). Then clearly \( A \mathbf{x} = \mathbf{b} \) has no solution.