SOLUTIONS

Exam 1, Mathematics 109
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Name:
Student ID:
Section Number:

Note: There are 3 problems on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted.

I. (40 points)

(1) Use truth tables or other well known tautologies (e.g. the de Morgan laws) to show that the following logical expression is a tautology.

$$(P \wedge \rceil Q) \vee (Q \wedge \rceil P) \longleftrightarrow (P \vee Q) \wedge \rceil (P \wedge Q)$$

(2) Use the tautology in (1) above to show that if A and B are two subsets of a universal set $\mathcal{U}$, then

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

(3) Compute $(A \setminus B) \cup (B \setminus A)$, for

$$A = \{x \mid x \in \mathbb{N}, \quad \exists k \in \mathbb{Z}, \quad x = 4k+1\}$$

$$B = \{x \mid x \in \mathbb{N}, \quad \exists \ell \in \mathbb{Z}, \quad x = 2\ell+1\}.$$

(1) $(P \wedge \rceil Q) \vee (Q \wedge \rceil P) \leftrightarrow ((P \wedge \rceil Q) \vee Q) \wedge ((P \wedge \rceil Q) \vee \rceil P) \leftrightarrow$

$\leftrightarrow ((P \vee Q) \wedge (\rceil Q \vee Q)) \wedge ((P \vee \rceil P) \wedge (\rceil Q \vee \rceil P)) \underset{*}{\leftrightarrow}$

$\underset{*}{\leftrightarrow} (P \vee Q) \wedge (\rceil Q \vee \rceil P) \leftrightarrow (P \vee Q) \wedge \rceil (P \wedge Q).$ $\square$

Equivalence "$\underset{*}{\leftrightarrow}$" above is based on the fact that $\rceil Q \vee Q$ and $\rceil P \vee P$ are tautologies.

(2) ~~I~~ Let $P(x): x \in A$; $Q(x): x \in B$; $\forall x \in \mathcal{U}$ (a universal set containing A and B.)

Then, by definition, (2) is equivalent to

$$\forall x \in \mathcal{U}, \ \left(P(x) \wedge \neg Q(x)\right) \vee \left(Q(x) \wedge \neg P(x)\right) \longleftrightarrow$$

$$\longleftrightarrow \left(P(x) \vee Q(x)\right) \wedge \neg\left(P(x) \wedge Q(x)\right).$$

The last statement is ~~true~~, as a direct consequence of the tautology (1). $\square$

(3) . $B \setminus A = \{x \mid x \in \mathbb{N}, \exists k \in \mathbb{Z}, \ x = 4k+3\}$ (prove this!)

Since $A \subseteq B$ (prove this!), $A \setminus B = \emptyset$.

Therefore

$$(B \setminus A) \cup (A \setminus B) = (B \setminus A) = \{x \mid x \in \mathbb{N}, \exists k \in \mathbb{Z}, \ x = 4k+3\}.$$

$\square$

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II. (30 points)

(1) Write formally the following statement: " For all real numbers x and y, such that $x < -4$ and $y > 2$, the distance between the point of coordinates (x, y) and the point of coordinates $(1, -2)$ is at least 6.

(2) Write the formal negation of the statement in (1) above.

(3) Prove or disprove the statement in (1) above and clearly indicate the method of proof used.

Hint: Please recall that the distance between $P(a, b)$ and $Q(c, d)$ is given by the formula $d(P, Q) = \sqrt{(a-c)^2 + (b-d)^2}$.

(1) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \ (x < -4) \wedge (y > 2) \rightarrow \sqrt{(x-1)^2 + (y+2)^2} \geq 6.$

(2) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \ \big((x < -4) \wedge (y > 2)\big) \wedge \left(\sqrt{(x-1)^2 + (y+2)^2} < 6\right)$

(3) We will prove the statement in (1).

Let $x, y \in \mathbb{R}$. We will show via direct proof that

$$(x < -4) \wedge (y > 2) \longrightarrow \sqrt{(x-1)^2 + (y+2)^2} \geq 6.$$

Assume that $(x < -4) \wedge (y > 2)$. Then

$$x - 1 < -4 - 1 = -5$$

$$y + 2 > 2 + 2 = 4$$

Therefore $(x-1)^2 > 25$, and $(y+2)^2 > 16$.

Therefore $\sqrt{(x-1)^2 + (y+2)^2} > \sqrt{25 + 16} =$

$$= \sqrt{41} > \sqrt{36} = 6$$

$\square$

III. (30 points)

(1) Write formally the following statement: "There exists a prime number p, for which there exists an integer x, such that $x^3 + x^2 + x = p - 1$."

(2) Prove or disprove the statement in (1) above.

Let $\mathcal{P} = \{p \in \mathbb{N} \mid p \text{ prime}\}$.

(1) $\exists p \in \mathcal{P}, \exists x \in \mathbb{Z}, \ x^3 + x^2 + x = p - 1.$

The negation of the statement above is:

$\forall p \in \mathcal{P}, \forall x \in \mathbb{Z}, \ x^3 + x^2 + x \neq p - 1 \qquad (*)$

(2) We will disprove the statement in (1), which is equivalent to proving its negation $(*)$.

Let $p \in \mathcal{P}$ and $x \in \mathbb{Z}$. Assume that $x^3 + x^2 + x = p - 1$. We will derive a <u>contradiction</u>.

$$x^3 + x^2 + x = p - 1 \implies x^3 + x^2 + x + 1 = p \implies$$

$$\implies (x+1)(x^2+1) = p. \qquad (**)$$

However, since p is prime, p has only two positive divisors, namely 1 and p. Therefore $(**)$ implies that:

$$x + 1 = 1 \quad \text{or} \quad x + 1 = p.$$

<u>Case I</u>. If $x + 1 = 1$, then $x = 0$. Therefore $(**)$ implies $1 = p$. This is a contradiction, as $1 \notin \mathcal{P}$.

Case 2

If $x+1=p$, then $x=p-1$. Therefore (*) implies that
$$p\cdot\left((p-1)^2+1\right)=p.$$

Consequently, $(p-1)^2+1=1$. Therefore $p-1=0$, Therefore $p=1$. This is a contradiction, as $1\notin\mathcal{P}$

$\square$