Problem 4.26: Let \( R = \{(a, b) \in \mathbb{R} \times \mathbb{R} : \text{there is an integer } k \text{ such that } a - b = 2k\pi\} \).

a) Prove that \( R \) is an equivalence relation on \( \mathbb{R} \).

**Proof.** To prove \( R \) is an equivalence relation, we must prove \( R \) is reflexive, symmetric, and transitive. So let \( a, b, c \in \mathbb{R} \). Then \( a - a = 0 = 0 \cdot 2\pi \) where \( 0 \in \mathbb{Z} \). Thus \( (a, a) \in R \) and \( R \) is reflexive.

Now suppose \( (a, b) \in R \). Then there exists \( k \in \mathbb{Z} \) such that \( a - b = 2k\pi \). Then \( b - a = -2k\pi = 2(-k)\pi \), and \( -k \in \mathbb{Z} \). Thus \( (b, a) \in R \) and \( R \) is symmetric.

If \( (a, b) \in R \) and \( (b, c) \in R \), then there exist integers \( k \) and \( n \) such that \( a - b = 2k\pi \) and \( b - c = 2n\pi \). Then

\[
a - c = (a - b) + (b - c) = 2k\pi + 2n\pi = 2(k + n)\pi
\]

where \( k + n \in \mathbb{Z} \). Thus \( (a, c) \in R \) and \( R \) is transitive. \( \blacksquare \)

b) List three members of \( \left[ \frac{\pi}{4} \right] \).

The elements of \( \left[ \frac{\pi}{4} \right] \) are real numbers \( b \) such that \( \frac{\pi}{4} - b = 2k\pi \) for some integer \( k \). That is, \( b = \frac{\pi}{4} - 2k\pi \) for some integer \( k \). Thus, we may generate elements of the equivalence class \( \left[ \frac{\pi}{4} \right] \) simply by plugging integers into the previous equation. For example, for \( k = 0, 1, 2 \), we have \( \frac{\pi}{4}, \frac{\pi}{4} - 2\pi = -\frac{7\pi}{4}, \) and \( \frac{\pi}{4} - 4\pi = -\frac{15\pi}{4} \) are elements of \( \left[ \frac{\pi}{4} \right] \).

c) List three members of \([1]\).

As in part b) above, we have 1, 1-2\( \pi \), and 1-4\( \pi \) are three members of \([1]\).

d) Which numbers, if any, belong to \( \left[ \frac{\pi}{4} \right] \cap [1] \)?

None. The intersection of these equivalence classes is empty. To see this, suppose there exists some \( x \in \left[ \frac{\pi}{4} \right] \cap [1] \). Then there exist some integers \( k \) and \( n \) such that \( \frac{\pi}{4} - 2k\pi = x \) and \( 1 - 2n\pi = x \). Thus,

\[
\frac{\pi}{4} - 2k\pi = 1 - 2n\pi.
\]

That is, \( k - n = \frac{1}{2\pi} \left( \frac{\pi}{4} - 1 \right) \). However, \( k - n \) is clearly an integer and the right hand side of this equation is not an integer. Thus, we have a contradiction, and there exists no such element \( x \) that belongs to both equivalence classes.
Problem 4.27: Let \( \mathbb{Q} \) be the set of all rational numbers, and let \( \mathbb{R} \) be the set of ordered pairs \((x, y)\) in \( \mathbb{Q} \times \mathbb{Q} \) such that when \( x \) and \( y \) are represented by fractions in lowest terms these fractions have the same denominator.

(a) Prove that \( \mathbb{R} \) is an equivalence relation on \( \mathbb{Q} \).

**Proof.** Let \( x, y, z \in \mathbb{Q} \). Then there exist some integers \( m, n, p, q, j, k \) such that \( \gcd(m, n) = 1 \), \( \gcd(p, q) = 1 \) and \( \gcd(j, k) = 1 \), and \( x = \frac{m}{n} \), \( y = \frac{pq}{j} \), and \( z = \frac{q}{k} \).

Certainly, \( n = n \) and so \((x, x) \in \mathbb{R}\). Thus \( \mathbb{R} \) is reflexive.

Suppose \((x, y) \in \mathbb{R}\). Then \( n = q \) implies \( q = n \) and so \((y, x) \in \mathbb{R}\).

Thus \( \mathbb{R} \) is symmetric.

Now suppose \((x, y) \in \mathbb{R}\) and \((y, z) \in \mathbb{R}\). Then \( n = q \) and \( q = k \) implies \( n = k \) and so \((x, z) \in \mathbb{R}\). Therefore \( \mathbb{R} \) is transitive. ■

(b) Prove that \([1/6] = [5/6]\).

**Proof.** \( y \in [1/6] \) iff there exists some integer \( n \) such that \( \gcd(n, 6) = 1 \) and \( y = n/6 \) iff \( y \in [5/6] \). ■

(c) Are \([4/6]\) and \([5/6]\) disjoint sets? Prove your answer.

Yes. \([4/6]\) and \([5/6]\) are disjoint because \(4/6\) and \(5/6\) have different denominators when represented in lowest terms.

Problem 4.36: For any two points \((a, b)\) and \((c, d)\) of the plane, define \((a, b) \cong (c, d)\) provided that \(a^2 + b^2 = c^2 + d^2\).

(a) Prove that \(\cong\) is an equivalence relation on \(\mathbb{R} \times \mathbb{R}\).

**Proof.** Let \((a, b), (c, d), (e, f) \in \mathbb{R} \times \mathbb{R}\). Clearly, \(a^2 + b^2 = a^2 + b^2\) and so \((a, b) \cong (a, b)\) and \(\cong\) is reflexive.

Now suppose \((a, b) \cong (c, d)\). Then \(a^2 + b^2 = c^2 + d^2\) implies \(c^2 + d^2 = a^2 + b^2\). So \((c, d) \cong (a, b)\) and \(\cong\) is symmetric.

If \((a, b) \cong (c, d)\) and \((c, d) \cong (e, f)\), then we have \(a^2 + b^2 = c^2 + d^2 = e^2 + f^2\) implies \((a, b) \cong (e, f)\) and \(\cong\) is transitive. ■

b) List all members of \([0, 0]\).

\([0, 0]\) = \{(0, 0)\} because there are no two non-zero real values \(a\) and \(b\) such that \(a^2 + b^2 = 0\).

c) Give a geometric description of \([5, 11]\).

\([5, 11]\) = \{ \((a, b) \in \mathbb{R} \times \mathbb{R} : a^2 + b^2 = 146\} \) is the set of all points on the circle with center at the origin and radius \(\sqrt{146}\).
Problem 4.45: For some \( n > 1 \), let \( S \) denote the set of all real \( n \times n \) matrices with real entries and let \( T \) denote the set of all invertible \( n \times n \) matrices. Define a relation \( \sim \) on \( S \) by \( A \sim B \) provided there is a matrix \( M \in T \) such that \( A = MBM^{-1} \). Prove that \( \sim \) is an equivalence relation on \( S \).

Proof. Let \( A, B, C \in S \). Note that the identity matrix \( I = I_n \) (the \( n \times n \) matrix with 1’s on the diagonal and 0’s everywhere else) is certainly invertible, and that \( IAI^{-1} = IAI = A \) implies \( A \sim A \). So \( \sim \) is reflexive on \( S \).

Now suppose \( A \sim B \). Then there exists \( M \in T \) such that \( A = MBM^{-1} \). Recall that for any invertible matrix \( M \), \( M^{-1} \) is also invertible and has inverse \((M^{-1})^{-1} = M \). Thus,

\[
M^{-1}AM = M^{-1}(MBM^{-1})M = (M^{-1}M)B(M^{-1}M) = B,
\]
and so \( B = M^{-1}AM \) with \( M^{-1} \in T \) implies \( B \sim A \). Thus, \( \sim \) is symmetric on \( S \).

If \( A \sim B \) and \( B \sim C \), then there exist invertible matrices \( M \) and \( N \) such that \( A = MBM^{-1} \) and \( B = NCN^{-1} \). Thus,

\[
A = MBM^{-1} = M( NCN^{-1})M^{-1} = (MN)C(MN)^{-1}
\]

since the product of two invertible matrices is invertible and \((MN)^{-1} = N^{-1}M^{-1} \). Thus, \( A \sim C \) and \( \sim \) is transitive. ■

Problem 4.62: Prove Theorem 4.8: If \( n \in \mathbb{N} \), congruence modulo \( n \) is an equivalence relation on the set of integers.

Proof. Let \( n \in \mathbb{N} \) and \( a, b, c \in \mathbb{Z} \). Then \( a - a = 0 \) and 0 is divisible by \( n \). So \( a \equiv a \mod n \), and the relation is reflexive.

Now suppose \( a \equiv b \mod n \). Then \( n|(a - b) \) so there exists an integer \( k \) such that \( a - b = kn \). Then \( b - a = -kn \) where \( -k \) is an integer, and so \( n|b - a \) and \( b \equiv a \mod n \). Thus, the relation is symmetric.

If \( a \equiv b \mod n \) and \( b \equiv c \mod n \), then \( n|(a - b) \) and \( n|(b - c) \). Note that \( a - c = (a - b) + (b - c) \) and so is the sum of two terms divisible by \( n \). Thus \( n|(a - c) \), and \( a \equiv c \mod n \) implies the relation is transitive. ■

Problem 4.64: For the equivalence relation \( a \equiv b \mod 9 \) we have that for each natural number \( n \), \([10^n] = [1] \).

Proof. Since the given relation is an equivalence relation, it suffices to show that for each natural number \( n \), \( 10^n \equiv 1 \mod 9 \).
Let $S = \{ n \in \mathbb{N} : 9 | (10^n - 1) \}$. $1 \in S$ because $9$ divides $10^1 - 1$. Assume $n \in S$. Then $9$ divides $10^n - 1$, and so there exists an integer $k$ such that $10^n - 1 = 9k$.

\[
10^{n+1} - 1 = 10 \cdot 10^n - 1 \\
= 9 \cdot 10^n + 10^n - 1 \\
= 9 \cdot 10^n + 9k \\
= 9(10^n + k),
\]

where the third inequality is due to the statement derived from the induction hypothesis. Hence $n + 1 \in S$. Therefore, by the Principle of Mathematical Induction, $S = \mathbb{N}$. ■